The zeros of Riemann's Zeta-Function on the critical line.

By

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1. Introduction.

1. We denote by $N_0(T)$ the number of zeros of $\zeta(s) = \zeta(\sigma + it)$ for which

$$\sigma = \tfrac{1}{2}, \quad 0 < t < T.$$

In a recent memoir in the *Acta Mathematica* [1]) we proved that the order of magnitude of $N_0(T)$ is not much less than $T^{\frac{3}{4}}$. More precisely, we proved that to every $\varepsilon > 0$ corresponds a $T_0 = T_0(\varepsilon)$ such that

$$N_0(T) > T^{\frac{3}{4} - \varepsilon} \qquad (T > T_0).$$

Here we go a good deal further. In § 2 we prove

Theorem A. *There is a $K > 0$ and a T_0 such that*

$$N_0(T) > KT \qquad (T > T_0). \tag{1.1}$$

The order of magnitude of $N_0(T)$ lies therefore between T and $T \log T$. In §§ 3—5 we prove, by rather more difficult analysis, a more precise result, viz.

Theorem B. *Let $U = T^a$, where $a > \frac{1}{2}$. Then there is a $K = K(a) > 0$ and a $T_0 = T_0(a)$ such that*

$$N_0(T + U) - N_0(T) > KU \qquad (T > T_0). \tag{1.2}$$

Some of the lemmas on which our argument depends have an interest independent of the particular application made of them here. We have therefore sometimes developed them further than is absolutely necessary for our immediate purpose.

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[1]) G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann Zeta-Function and the theory of the distribution of primes, *Acta Mathematica,* **41** (1917), 119–196 (177–184).

2. Proof that $N_0(T) > KT$.

2.1. Lemma 1[2]). *If* $\sigma > 0$, $s \neq 1$, $x > 0$, *then*

$$(2.11) \qquad \zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n \leq x}{}' n^{-s} = 2(2\pi)^{s-1} \sum_{n=1}^{\infty} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s} \cos u \, du,$$

or

$$(2.111) \qquad \zeta(s) + \frac{x^{1-s}}{1-s} + \left(\tfrac{1}{2} - x + [x]\right) x^{-s} - \sum_{n<x} n^{-s}$$

$$= 2s(2\pi)^{s-1} \sum_{n=1}^{\infty} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s-1} \sin u \, du.$$

Suppose first that $\sigma > 1$; and let

$$\psi(v) = \tfrac{1}{2} - v + [v], \qquad X = [x].$$

Then

$$(2.12) \qquad s \int_x^{\infty} v^{-s-1} \psi(v) \, dv = \frac{1}{2} x^{-s} - \frac{s x^{1-s}}{s-1} + s \int_x^{\infty} v^{-s-1} [v] \, dv$$

$$= \frac{1}{2} x^{-s} - \frac{s x^{1-s}}{s-1} + X\left(x^{-s} - (X+1)^{-s}\right) + \sum_{n=X+1}^{\infty} n\left(n^{-s} - (n+1)^{-s}\right).$$

Also

$$(2.13) \quad \zeta(s) - \sum_{n \leq x} n^{-s} = \sum_{n=X+1}^{\infty} n^{-s} = -X(X+1)^{-s} + \sum_{n=X+1}^{\infty} n\left(n^{-s} - (n+1)^{-s}\right).$$

From (2.12) and (2.13) it follows that

$$(2.14) \quad \zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n<x} n^{-s} = -x^{-s} \psi(x) + s \int_x^{\infty} v^{-s-1} \psi(v) \, dv.$$

Now

$$(2.15) \qquad \psi(v) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi v}{n},$$

except when v is an integer, when the sum of the series is 0. The trigonometrical series is boundedly convergent throughout any interval of values of v, and $\int_2^{\infty} |v^{-s-1}| \, dv$ is convergent. Hence[3]) we may multiply

[2]) The dash over the sign of summation indicates that, if x is an integer, the last term x^{-s} is to be replaced by $\frac{1}{2} x^{-s}$.

The lemma may be proved in various ways. The method followed here was suggested to us by Dr. H. Cramér of Stockholm, and is materially simpler than that which we had adopted originally.

[3]) See, for example, W. H. Young, The application of expansions to definite integrals, *Proc. London Math. Soc.* (2), 9 (1919), 463—485 (468).

(2.15) by $s v^{-s-1}$ and integrate term by term over the interval (x, ∞). Thus

$$s\int_x^\infty v^{-s-1}\psi(v)\,dv = \frac{s}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_x^\infty v^{-s-1}\sin 2n\pi v\,dv$$

$$= 2s(2\pi)^{s-1}\sum_{n=1}^\infty n^{s-1}\int_{2n\pi x}^\infty u^{-s-1}\sin u\,du,$$

(2.16) $$\zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n<x} n^{-s} = -x^{-s}\psi(x)$$

$$+ 2s(2\pi)^{s-1}\sum_{n=1}^\infty n^{s-1}\int_{2n\pi x}^\infty u^{-s-1}\sin u\,du,$$

which is equivalent to (2.111). This equation is so far proved only when $\sigma > 1$. But

$$n^{s-1}\int_{2n\pi x}^\infty u^{-s-1}\sin u\,du = n^{s-1}(2n\pi x)^{-s-1}\cos 2n\pi x - (s+1)\,n^{s-1}\int_{2n\pi x}^\infty u^{-s-2}\cos u\,du$$

$$= O(n^{-2}) + n^{s-1}\int_{2n\pi x}^\infty O(u^{-\sigma-2})\,du = O(n^{-2}),$$

provided only $\sigma > -1$, and the series in (2.16) is uniformly convergent for $\sigma \geqq -1+\delta > -1$ and any finite range of t. Hence (2.16) is valid for $\sigma > -1$. Also

$$2s(2\pi)^{s-1} n^{s-1}\int_{2n\pi x}^\infty u^{-s-1}\sin u\,du = \frac{x^{-s}}{\pi}\frac{\sin 2n\pi x}{n} + 2(2\pi)^{s-1} n^{s-1}\int_{2n\pi x}^\infty u^{-s}\cos u\,du$$

if $\sigma > 0$. Substituting in (2.16), we obtain (2.11).

The equation (2.111) holds for the wider region $\sigma > -1$. If we suppose $-1 < \sigma < 0$, and make x tend to zero, we obtain the classical functional equation. The equations are easily modified so as to yield representations for $\zeta(s)$ valid over an arbitrary half-plane.

2.2. Lemma 2. *Suppose that* $\sigma \geqq \sigma_0 > 0$, $|s-1| \geqq \delta > 0$, *and*

(2.21) $$|t| < \frac{2\pi x}{C},$$

where $C > 1$. *Then*

(2.22) $$\zeta(s) = \sum_{n \leqq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

uniformly in s.

We use the result of Lemma 1 in the form (2.111). It is plainly sufficient to prove that

(2.23) $$s\sum n^{s-1} I_n = s\sum n^{s-1}\int_{2n\pi x}^\infty u^{-s-1}\sin u\,du = O(x^{-\sigma}).$$

We have

$$2iI_n = \int_{2n\pi x}^{\infty} u^{-s-1} e^{iu}\,du - \int_{2n\pi x}^{\infty} u^{-s-1} e^{-iu}\,du = j_n + j'_n,$$

say. Also

$$j_n = \int_{2n\pi x}^{\infty} u^{-s-1} e^{iu}\,du = \int_{2n\pi x}^{\infty} u^{-\sigma-1} e^{i(u-t\log u)}\,du = \int_{2n\pi x}^{\infty} u^{-\sigma-1} e^{iw}\,du,$$

where $w = u - t\log u$. Now

$$\frac{dw}{du} = 1 - \frac{t}{u} \geqq 1 - \frac{t}{2n\pi x} \geqq 1 - \frac{t}{2\pi x} > 1 - \frac{1}{C}$$

and increases steadily as u increases from $2n\pi x$ to infinity. Hence

$$\Re(j_n) = \int_{u=2n\pi x}^{\infty} u^{-\sigma-1} \frac{\cos w\,dw}{1-\frac{t}{u}} = O(nx)^{-\sigma-1} \int_{w_1}^{w_2} \cos w\,dw = O(nx)^{-\sigma-1}.$$

The same argument may be applied to the imaginary part of j_n and to both components of j'_n [4]). Hence

$$I_n = O(nx)^{-\sigma-1},$$

and the series (2.23) takes the form

$$O\left(|t|\sum n^{\sigma-1} O(nx)^{-\sigma-1}\right) = O(x^{-\sigma-1}|t|) = O(x^{-\sigma}).$$

Lemma 3. *There is a constant A such that*

$$\eta(s) = (1-2^{1-s})\,\zeta(s) = \sum_{n\leqq x} (-1)^{n-1} n^{-s} + O(x^{-\sigma})$$

uniformly for $\sigma \geqq \sigma_0 > 0$, $|t| < Ax$.

For

$$\sum_{n\leqq x} (-1)^{n-1} n^{-s} = \sum_{n\leqq x} n^{-s} - 2^{1-s} \sum_{n\leqq \frac{1}{2}x} n^{-s} + O(x^{-\sigma})$$

$$= (1-2^{1-s})\,\zeta(s) + \frac{x^{1-s}}{1-s} - 2^{1-s}\frac{(\frac{1}{2}x)^{1-s}}{1-s} + O(x^{-\sigma}) = \eta(s) + O(x^{-\sigma})$$

by Lemma 2, if A is sufficiently small. The restriction that $|s-1| \geqq \delta > 0$ may obviously be omitted here.

Lemma 4. *There is a constant A such that*

$$\Theta(s) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n^{-s}}{\log n} = \sum_{n\leqq x} \frac{(-1)^{n-1} n^{-s}}{\log n} + O(x^{-\sigma})$$

uniformly for $\sigma \geqq \sigma_0 > 0$, $|t| < Ax$.

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[4]) The argument with j'_n is simpler, as $1+\frac{t}{u}$ occurs instead of $1-\frac{t}{u}$, and the inequality (2.21) is not required.

For, if $[x] = X$,

$$\sum_{n=X+1}^{\infty} \frac{(-1)^{n-1} n^{-s}}{\log n} = \sum_{n=X+1}^{\infty} \left(\frac{1}{\log n} - \frac{1}{\log (n+1)}\right) \sum_{m=X+1}^{n} (-1)^{n-1} n^{-s}$$

$$= \sum_{n=X+1}^{\infty} O\left(\frac{1}{n(\log n)^2}\right) O(x^{-\sigma}) = O(x^{-\sigma}),$$

by Lemma 3.

2.3. Lemma 5. *If* $1 \leqq m \leqq \mu$, $1 \leqq n \leqq \mu$, $m \neq n$, *then*

(2.31) $$\sum \frac{1}{\sqrt{mn} \left|\log \frac{m}{n}\right|} = O(\mu \log \mu).$$

We write

$$\sum = \sum_{m < \frac{1}{2}n} + \sum_{\frac{1}{2}n \leqq m \leqq \frac{3}{2}n} + \sum_{\frac{3}{2}n < m} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

say. Then

$$\sum\nolimits_1 = O\left(\sum_{m,n=1}^{\mu} \frac{1}{\sqrt{mn}}\right) = O\left(\sum_{n=1}^{\mu} \frac{1}{\sqrt{n}}\right)^2 = O(\mu),$$

and so for $\sum_3$. In $\sum_2$ we have $m = n + r$, where $|r| \leqq \frac{1}{2}n$, and

$$\frac{1}{\left|\log \frac{m}{n}\right|} = O\left(\frac{n}{|r|}\right).$$

Hence

$$\sum\nolimits_2 = O\left(\sum_{n=1}^{\mu} \sum_{r=1}^{\frac{1}{2}n} \frac{1}{\sqrt{n(n+r)}} \frac{n}{r}\right) = O\left(\sum_{n=1}^{\mu} \sum_{r=1}^{\mu} \frac{1}{r}\right) = O(\mu \log \mu).$$

This lemma is frequently useful. But what we shall require immediately is a slightly different result, viz:

Lemma 6. *If* $2 \leqq m \leqq \mu$, $2 \leqq n \leqq \mu$, $m \neq n$, *then*

(2.32) $$\sum \frac{1}{\sqrt{mn} \log m \log n \left|\log \frac{m}{n}\right|} = O\left(\frac{\mu}{\log \mu}\right).$$

Dividing up the summation as in Lemma 5, we obtain

$$\sum\nolimits_1 = O\left(\sum_{n=2}^{\mu} \frac{1}{\sqrt{n} \log n}\right)^2 = O\left(\frac{\mu}{(\log \mu)^2}\right)$$

and

$$\sum\nolimits_2 = O\left(\sum_{n=2}^{\mu} \frac{1}{(\log n)^2} \sum_{r=1}^{\mu} \frac{1}{r}\right) = O\left(\frac{\mu}{\log \mu}\right).$$

2.4. Lemma 7. *If*

(2.41) $$\psi(t) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\log n} n^{-\frac{1}{2} - it}$$

then

$$\int_T^{2T} |\psi(t+u)|^2 dt = O(T) \tag{2.42}$$

uniformly for $0 \leqq u \leqq T$.

This is the proposition we shall actually use. It is plainly sufficient to prove it when $u = 0$; and in this form it is an immediate corollary of

Lemma 8. *If*

$$\chi(s) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n^{-s}}{\log n} \tag{2.43}$$

then

$$\frac{1}{2T}\int_{-T}^{T} \left|\chi\left(\tfrac{1}{2}+it\right)\right|^2 dt \sim \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}. \tag{2.44}$$

By Lemma 4, we have

$$\chi\left(\tfrac{1}{2}+it\right) = \sum_{n<AT} \frac{(-1)^{n-1}}{\log n} n^{-\frac{1}{2}-it} + O(T^{-\frac{1}{2}}) = \Theta + O(T^{-\frac{1}{2}}), \tag{2.45}$$

say; and so

$$\begin{aligned} \int_{-T}^{T} \left|\chi\left(\tfrac{1}{2}+it\right)\right|^2 dt &= \int_{-T}^{T} |\Theta|^2 dt + O\left(T^{-\frac{1}{2}}\int_{-T}^{T}|\Theta|\,dt\right) + O(1) \\ &= \int_{-T}^{T} |\Theta|^2 dt + O\left(\int_{-T}^{T}|\Theta|^2 dt\right)^{\frac{1}{2}} + O(1). \end{aligned} \tag{2.46}$$

Now

$$\begin{aligned} \int_{-T}^{T} |\Theta|^2 dt &= 2T \sum_{n<AT} \frac{1}{n(\log n)^2} + \sum \frac{(-1)^{m+n}}{\sqrt{mn}\log m \log n} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt \\ &= 2T \sum_{n<AT} \frac{1}{n(\log n)^2} + O\left(\sum \frac{1}{\sqrt{mn}\log m \log n \left|\log \frac{m}{n}\right|}\right), \end{aligned} \tag{2.47}$$

where the double summations are defined as in Lemma 6, with $\mu = AT$. From (2.32), (2.47), and (2.46) it follows that

$$\int_{-T}^{T} \left|\chi\left(\tfrac{1}{2}+it\right)\right|^2 dt \sim \int_{-T}^{T} |\Theta|^2 dt \sim 2T \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}.$$

2.5. Lemma 9. *If* $0 < k < 1$, *the real parts*[5]) *of a and b lie between*

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[5]) We are concerned with the case in which a and b are pure imaginaries. For the definition of $H_1^{\nu}(x)$, see Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, p. 17.

$-k$ and $1-k$, the real part of y is positive, and y^{-2s} has its principal value, then

$$(2.51)\qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\Gamma(s+a)\,\Gamma(s+b)\,y^{-2s}\,ds = i\pi y^{a+b}e^{\frac{1}{2}\nu\pi i}H_1^{\nu}(2iy),$$

where

$$(2.511)\qquad \nu = a-b$$

and $H_1^{\nu}(x)$ is Hankel's cylinder-function.

We find, in fact, by a straightforward calculation of which it is hardly necessary to give the details, that the value of the integral is

$$\frac{\pi y^{a+b}}{\sin(a-b)\pi}\left(e^{\frac{1}{2}(a-b)\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n(iy)^{2n-a+b}}{n!\,\Gamma(1-a+b+n)} - e^{-\frac{1}{2}(a-b)\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n(iy)^{2n+a-b}}{n!\,\Gamma(1+a-b+n)}\right)$$

$$= -\frac{\pi y^{a+b}}{\sin\nu\pi}e^{\frac{1}{2}\nu\pi i}\left(e^{-\nu\pi i}J_{\nu}(2iy) - J_{-\nu}(2iy)\right)$$

$$= i\pi y^{a+b}e^{\frac{1}{2}\nu\pi i}H_1^{\nu}(2iy).$$

This proof supposes that $a \neq b$. The result may be at once extended to cover this case by a passage to the limit.

Lemma 10. *If $k > \frac{1}{2}$, and the other conditions of Lemma 9 are satisfied, then*

$$(2.52)\qquad \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\Gamma(s+a)\,\Gamma(s+b)\,\zeta(2s+2a)\,\zeta(2s+2b)\,y^{-2s}ds$$

$$= i\pi e^{\frac{1}{2}\nu\pi i}y^{a+b}\sum_{n=1}^{\infty}c_n H_1^{\nu}(2iny),$$

where

$$c_n = n^{b-a}\sigma_{2(a-b)}(n) = n^{-\nu}\sigma_{2\nu}(n) = n^{\nu}\sigma_{-2\nu}(n),$$

and $\sigma_r(n)$ denotes the sum of the r-th powers of the divisors of n.

We have

$$(2.53)\qquad \zeta(2s+2a)\,\zeta(2s+2b) = \sum\frac{d_n}{n^{2s}} \qquad \left(\sigma > \frac{1}{2}\right),$$

where

$$(2.531)\qquad d_n = \sum_{d|n}d^{-2a}\left(\frac{n}{d}\right)^{-2b} = n^{-2a}\sigma_{2\nu}(n) = n^{-a-b}c_n.$$

If now we write ny for y in (2.51), multiply by d_n, and sum, we obtain (2.52).

2.6. Suppose that $s = \frac{1}{2}+it$, and write

$$\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \frac{2\pi^{\frac{1}{2}s}\Xi(t)}{s(s-1)} = -\frac{2\pi^{\frac{1}{4}+\frac{1}{2}it}\Xi(t)}{\frac{1}{4}+t^2} = 2\pi^{\frac{1}{4}}t^{-\frac{7}{4}}e^{-\frac{1}{4}\pi t}\pi^{\frac{1}{2}it}X(t),$$

so that $X(t)$ is real for real t. Supposing t positive, and approximating to the Gamma-function by Stirling's Theorem, we obtain

$$(2.61)\qquad \zeta(s) = -\left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{\frac{1}{8}\pi i}(2\pi e)^{\frac{1}{2}it} e^{-\frac{1}{2}it\log t} X(t)\left(1+O\left(\frac{1}{t}\right)\right).$$

There is of course a conjugate formula when $t < 0$.

We write

$$(2.62)\qquad I = I(t, H) = \int_t^{t+H} X(u)\,du.$$

Here H is a constant, which will ultimately be chosen large enough to satisfy certain conditions. We shall suppose $H > 2$.

In the arguments preceding 4. 2 A denotes generally an absolute positive constant; so also do B, C, ... A few words are necessary as to the use of O. The constants implied by the O's will also be absolute; but there is a reservation which must be made as to the values of the variables (t, T, ε, n, m, ...) for which the inequalities symbolised by the O's are satisfied. We shall frequently be concerned with inequalities of the type (*e. g.*)

$$(2.63)\qquad |F(t)| < f(H)\varphi(t),$$

and, if we wrote this simply in the form $F = O(\varphi)$, the constant of the O would depend upon H. If $f(H)$ is a simple function of H (*e. g.* H), we may write

$$F = O(H\varphi) \qquad (i.\ e.\ |F| < AH\varphi),$$

but sometimes it would be troublesome to maintain this degree of explicitness. We shall therefore sometimes write

$$F = O(f(H)\varphi)$$

meaning thereby that (2. 63) is satisfied for *some* form of the function $f(H)$.

The choice of H will always be prior logically to that of the variables $t, T, \ldots$ which tend to limits. We shall therefore have

$$o(f(H)\varphi) + o(f_1(H)\varphi_1) = o(f(H)\varphi)$$

if $\varphi_1 = O(\varphi)$, whatever be the forms of f and f_1. We can extend this principle to O, writing, *e. g.*,

$$O(T\sqrt{H}) + O(H\sqrt{T}) = O(T\sqrt{H})$$

(since $H\sqrt{T} < T\sqrt{H}$ for $T > T_0(H)$). But then it must be understood that the inequalities symbolised by the O's are only satisfied when T exceeds a certain value, depending on H alone. As we shall, in such cases, be concerned with large values of T only, and H is chosen first, there is no inconvenience in this reservation.

From 4. 2 onwards $A, B, C, \ldots$ denote positive numbers depending only on the a of Theorem B. and the constants implied by the O's depend upon a only.

2.71. Lemma 11. *We have*

(2.711) $$\int_0^T \{I(t, H)\}^2 dt < AHT \qquad (T > T_0 = T_0(H)).$$

We note first that it is sufficient to prove

(2.712) $$\int_0^\infty e^{-\varepsilon t} I^2 dt < \frac{AH}{\varepsilon} \qquad (0 < \varepsilon < \varepsilon_0 = \varepsilon_0(H)).$$

For then

$$\int_0^T I^2 dt < e\int_0^T e^{-\frac{t}{T}} I^2 dt < e\int_0^\infty e^{-\frac{t}{T}} I^2 dt < AHT \qquad (T > T_0).$$

2.72. We return to the result of Lemma 10, taking

$$a = i\alpha, \quad b = i\beta, \quad 0 \leqq \alpha \leqq H, \quad 0 \leqq \beta \leqq H, \quad \alpha \neq \beta.$$

It follows from (2.52) and Cauchy's Theorem[6]) that

(2.721) $$J = \frac{1}{2\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)y^{-2s}ds$$

$$= i\pi e^{\frac{1}{2}\nu\pi i} y^{a+b}\sum_{n=1}^\infty c_n H_1^\nu(2iny) + \Phi,$$

where

$$\Phi = \tfrac{1}{2}\sqrt{\pi}\{\Gamma(\tfrac{1}{2}+b-a)\zeta(1+2b-2a)y^{-1+2a} + \Gamma(\tfrac{1}{2}+a-b)\zeta(1+2a-2b)y^{-1+2b}\}.$$

2.73. We take

(2.731) $$y = \pi e^{i\theta} = \pi e^{i(\frac{1}{2}\pi-\varepsilon)} \qquad (\varepsilon > 0)$$

and make $\varepsilon \to 0$. It is plain, first that $|\Phi| < f(H)$ or

(2.732) $$\Phi = O(f(H)).$$

Next, we write

(2.733) $$J = \frac{1}{2\pi i}\left(\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}-iH} + \int_{\frac{1}{4}-iH}^{\frac{1}{4}} + \int_{\frac{1}{4}}^{\frac{1}{4}+i\infty}\right) = J_1 + J_2 + J_3.$$

[6]) If $\arg y = \theta$, so that $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, we have

$$\Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)y^{-2s} = O(|t|^A e^{-(\pi-2\theta)|t|}),$$

so that the deformation of the contour presents no difficulty.

Here it is only J_3 that is of importance[7]). We have plainly

(2.734) $$J_2 = O(f(H)).$$

In J_1

$$|\Gamma(\tfrac{1}{4}+i(t+\alpha))| < A|t+\alpha|^{-\frac{1}{4}} e^{-\frac{1}{2}\pi|t+\alpha|} < A|t+H|^{-\frac{1}{4}} e^{-\frac{1}{2}\pi|t+H|},$$
$$|\zeta(\tfrac{1}{2}+2i(t+\alpha))| < A(|t|^{\frac{1}{3}}+1),$$

and similarly for the factors involving β; and

$$|y^{-2s}| < A\, e^{2\theta t} < A\, e^{-\frac{1}{2}\pi|t|} \qquad (\varepsilon < \tfrac{1}{4}\pi).$$

Hence

(2.735) $$|J_1| < A\int_{-\infty}^{-H} |t+H|^{-\frac{1}{2}}(|t|^{\frac{1}{3}}+1)^2 e^{-\pi|t+H|-\frac{1}{2}\pi|t|}\,dt$$
$$< A\int_0^{\infty} t^{-\frac{1}{2}}(|t+H|^{\frac{1}{3}}+1)^2 e^{-\pi t-\frac{1}{2}\pi(t+H)}\,dt < f(H).$$

We have therefore, from (2.721), (2.732), (2.733), (2.734), and (2.735)

(2.736) $$J_3 = \frac{1}{2\pi i}\int_{\frac{1}{4}}^{\frac{1}{4}+i\infty} = i\pi e^{\frac{1}{2}\nu\pi i} y^{a+b} \sum_{n=1}^{\infty} c_n H_1^{\nu}(2iny) + O(f(H)).$$

2.74. In J_3 we have

$$y^{-2s} = \pi^{-\frac{1}{2}-2it} e^{-\frac{1}{4}(\pi-2\varepsilon)i} e^{(\pi-2\varepsilon)t},$$

$$\Gamma(s+a)\zeta(2s+2a) = \Gamma(\tfrac{1}{4}+i(t+\alpha))\zeta(\tfrac{1}{2}+2i(t+\alpha))$$
$$= A(t+\alpha)^{-\frac{1}{4}} e^{-\frac{1}{2}\pi(t+\alpha)} \pi^{i(t+\alpha)} X(2t+2\alpha)\left(1+O\left(\frac{1}{t}\right)\right),$$
$$= A\, t^{-\frac{1}{4}} e^{-\frac{1}{2}\pi(t+\alpha)} \pi^{i(t+\alpha)} X(2t+2\alpha)\left(1+O\left(\frac{H}{t}\right)\right),$$

$$\Gamma(s+a)\Gamma(s+b)\zeta(2s+2a)\zeta(2s+2b)$$
$$= A\, t^{-\frac{1}{2}} e^{-\pi t-\frac{1}{2}\pi(\alpha+\beta)} \pi^{2it+i(\alpha+\beta)} X(2t+2\alpha)X(2t+2\beta)\left(1+O\left(\frac{H}{t}\right)\right).$$

Since $X(2t+2\alpha)X(2t+2\beta) = O(f(H)t^{\frac{2}{3}})$ and $y^{-2s} = O(e^{(\pi-2\varepsilon)t})$, the error term contributes

$$O\left(f(H)\int_0^{\infty} t^{-\frac{5}{6}} e^{-2\varepsilon t}\,dt\right) = O(f(H)\varepsilon^{-\frac{1}{6}}).$$

We have therefore

(2.741) $$J_3 = A\, e^{-\frac{1}{4}\pi i} e^{-\frac{1}{2}\pi(\alpha+\beta)} \pi^{i(\alpha+\beta)} (1+O(\varepsilon)) \int_0^{\infty} t^{-\frac{1}{2}} X(2t+2\alpha)X(2t+2\beta) e^{-}$$
$$+ O(f(H)\varepsilon^{-\frac{1}{6}}).$$

---

[7]) Because $|y^{-2s}| < A e^{2\theta t}$ and the Gamma-functions provide a factor $e^{-\pi|t|}$.

2.75. Turning our attention to the series on the right hand side of (2.736), we have

$$i\pi e^{\frac12\nu\pi i} = A\, i\, e^{-\frac12\pi(\alpha-\beta)},$$

$$y = \pi e^{(\frac12\pi-\varepsilon)i} = i\pi + \varepsilon + O(\varepsilon^2),$$

$$y^{a+b} = y^{i(\alpha+\beta)} = \pi^{i(\alpha+\beta)} e^{-\frac12\pi(\alpha+\beta)}(1+O(\varepsilon)),$$

$$H_1^{\nu}(2iny) = \frac{A}{\sqrt{nyi}} e^{-2ny-\frac14\pi i(2\nu+1)}\left(1+O\left(\frac1n\right)\right) \text{ [8]}$$

$$= \frac{A}{\sqrt{i}}(1+O(\varepsilon))\, e^{\frac12\pi(\alpha-\beta)}\, \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}}\, e^{O(n\varepsilon^2)}\left(1+O\left(\frac1n\right)\right).$$

(2.751) $$i\pi e^{\frac12\nu\pi i} y^{a+b}\sum_{n=1}^{\infty} c_n H_1^{\nu}(2iny)$$

$$= \frac{A}{\sqrt{i}} e^{-\frac12\pi(\alpha+\beta)}\pi^{i(\alpha+\beta)}(1+O(\varepsilon))\sum_{n=1}^{\infty} c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} e^{O(n\varepsilon^2)}\left(1+O\left(\frac1n\right)\right).$$

Here we may replace $e^{O(n\varepsilon^2)}$ by $1+O(n\varepsilon^2)$, since $e^{An\varepsilon^2} < 1 + An\varepsilon^2$ if $n\varepsilon^2 < A$ and

$$\sum_{n\varepsilon^2 > A} |c_n| \frac{e^{-2n\pi\varepsilon + O(n\varepsilon^2)}}{\sqrt{n}} < \sum_{n\varepsilon^2 > A} n e^{-n\pi\varepsilon} < A e^{-\frac{A}{\varepsilon}}.$$

Making this simplification, and comparing (2.736), (2.741) and (2.751), we obtain

(2.752) $$(1+O(\varepsilon))\int_0^{\infty} t^{-\frac12} X(2t+2\alpha) X(2t+2\beta) e^{-2\varepsilon t} dt + O(f(H)\varepsilon^{-\frac16})$$

$$= (1+O(\varepsilon)) A \sum_{n=1}^{\infty} c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}}\left(1+O\left(\frac1n\right)+O(n\varepsilon^2)\right) + O(f(H)).$$

Now

$$c_n = n^{i(\beta-\alpha)}\sigma_{2i(\alpha-\beta)}(n) = O(d(n)) = O\left(n^{\frac14}\right),$$

$$\sum c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} O\left(\frac1n\right) = O\left(\sum n^{-\frac54} e^{-2n\pi\varepsilon}\right) = O(1),$$

$$\sum c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} O(n\varepsilon^2) = O\left(\varepsilon^2 \sum n^{\frac34} e^{-2n\pi\varepsilon}\right) = O(1),$$

$$O(\varepsilon)\sum c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} = O\left(\varepsilon \sum n^{-\frac14} e^{-2n\pi\varepsilon}\right) = O(1).$$

Hence (2.752) may be written in the simpler form

(2.753) $$\int_0^{\infty} t^{-\frac12} X(2t+2\alpha) X(2t+2\beta) e^{-2\varepsilon t} dt = A\sum_{n=1}^{\infty} c_n \frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} + O\left(f(H)\varepsilon^{-\frac16}\right).$$

---

[8] Nielsen, loc. cit., p. 154.

2.76. We now integrate (2.753) with respect to α and β, in each case over the interval $(0, H)$. So long as ε is positive, the series and integral are uniformly convergent, and we may invert the orders of integration and summation. Since

$$\int_0^H X(2t+2\alpha)\,d\alpha = \frac{1}{2}\int_{2t}^{2(t+H)} X(u)\,du,$$

the left hand side gives

$$\int_0^\infty t^{-\frac{1}{2}} e^{-2\varepsilon t}\,dt \int_0^H X(2t+2\alpha)\,d\alpha \int_0^H X(2t+2\beta)\,d\beta = \frac{1}{4}\int_0^\infty t^{-\frac{1}{2}} e^{-2\varepsilon t}(I(2t, 2H))^2\,dt.$$

Thus, if we write

$$(2.761) \qquad C_n = \int_0^H\int_0^H c_n\,d\alpha\,d\beta = \int_0^H\int_0^H n^{i(\alpha-\beta)}\sum_{d|n} d^{2i(\beta-\alpha)}\,d\alpha\,d\beta$$

$$= \sum_{d|n}\int_0^H\int_0^H \left(\frac{n}{d^2}\right)^{i(\alpha-\beta)} d\alpha\,d\beta = A\sum_{d|n}\left(\frac{\sin\left(\frac{1}{2}H\log\frac{n}{d^2}\right)}{\log\frac{n}{d^2}}\right)^2,$$

we obtain

$$(2.762) \qquad \int_0^\infty t^{-\frac{1}{2}} e^{-2\varepsilon t}(I(2t, 2H))^2\,dt = A\sum_{n=1}^\infty C_n\frac{e^{-2n\pi\varepsilon}}{\sqrt{n}} + O\left(f(H)\,\varepsilon^{-\frac{1}{2}}\right).$$

2.77. We proceed to consider the sum

$$(2.771) \qquad \underset{\sim}{C}_m = C_1 + C_2 + \ldots + C_m = \sum_{x>0,\, y>0,\, xy\leqq m}\left(\frac{\sin\left(\frac{1}{2}H\log\frac{x}{y}\right)}{\log\frac{x}{y}}\right)^2$$

$\left(\text{since } \frac{n}{d^2} = \frac{x}{y} \text{ if } n = xy \text{ and } d = y\right)$. We can write

$$(2.772) \qquad \underset{\sim}{C}_m \leqq 2\left(\sum\nolimits_1 + \sum\nolimits_2\right),$$

where $\sum_1$ is defined by

$$0 < kx \leqq y \leqq x, \quad xy \leqq m,$$

and $\sum_2$ by

$$0 < y \leqq kx, \quad xy \leqq m.$$

Here $\frac{1}{2} < k < 1$: we shall ultimately take $k = 1 - \frac{1}{H}$.

2.781. In $\sum_1$ we use the inequality

$$\left(\frac{\sin Hu}{u}\right)^2 < H^2.$$

We obtain

$$(2.7811)\quad \sum\nolimits_1 \leqq H^2\Big(\sum_{1\leqq x\leqq\sqrt{m}}((1-k)x+1)+\sum_{\sqrt{m}\leqq x\leqq\sqrt{\frac{m}{k}}}\Big(\frac{m}{x}-kx+1\Big)\Big)$$

$$=O(H^2 m(1-k))+O(H^2\sqrt{m})+O\Big(H^2 m\log\frac{1}{k}\Big)$$

$$=O\Big(H^2 m\log\frac{1}{k}\Big)+O(H^2\sqrt{m}),$$

when $m\to\infty$.

2.782. The terms of $\sum_2$ we divide into two classes as follows. Associate with the point (x,y) the square $Q_{x,y}$ of which two opposite corners are (x,y) and $(x-1,\ y+1)$. In the first class γ_1 we put all terms (x,y) for which the associated square does not cross the line $y=kx$; in the second class γ_2 the terms for which it crosses the line.

It is plain that, if (x,y) belongs to γ_1,

$$\frac{1}{\left(\log\frac{x}{y}\right)^2}<\iint\limits_{Q(x,y)}\frac{d\xi\,d\eta}{\left(\log\frac{\xi}{\eta}\right)^2}.$$

Hence

$$\sum_{\gamma_1}<\iint\frac{d\xi\,d\eta}{\left(\log\frac{\xi}{\eta}\right)^2},$$

when the domain of integration is defined by $0\leqq\eta\leqq k\xi,\ \xi\leqq m,\ \xi(\eta-1)\leqq m$, and *a fortiori* when it is defined by $0\leqq\eta\leqq k\xi,\ \xi\eta\leqq 2m$. Transforming to polar coordinates, we obtain

$$(2.7821)\quad \sum_{\gamma_1}<\int_0^{\operatorname{arc\,tan}k}\frac{d\theta}{(\log\tan\theta)^2}\int_0^{\sqrt{\frac{2m}{\cos\theta\sin\theta}}}r\,dr=2m\int_0^{\operatorname{arc\,tan}k}\frac{d\theta}{\cos\theta\sin\theta\,(\log\tan\theta)^2}$$

$$=2m\int_0^k\frac{dt}{t(\log t)^2}=O\left(\frac{m}{\log\frac{1}{k}}\right).$$

The number of terms of γ_2 is less than a constant multiple of the length of the line joining the origin to the point $\left(\sqrt{\frac{m}{k}},\sqrt{km}\right)$, or of $\sqrt{m}$. Hence

$$(2.7822)\qquad \sum_{\gamma_2}=O(H^2\sqrt{m}).$$

From (2.772), (2.7811), (2.7821), and (2.7822) we obtain

$$\underset{\sim}{C}_m=O\Big(H^2 m\log\frac{1}{k}\Big)+O(H^2\sqrt{m})+O\left(\frac{m}{\log\frac{1}{k}}\right).$$

Taking now $k = 1 - \frac{1}{H}$, we obtain

(2.783) $$\underset{\sim}{C}_m = O(Hm) + O(H^2\sqrt{m}) = O(Hm).$$

2.79. We can now complete the proof of Lemma 11. We have, by (2.783) and partial summation,

$$\sum_{n=1}^{m} \frac{C_n}{\sqrt{n}} = O(H\sqrt{m})$$

and so

$$\sum_{n=1}^{\infty} \frac{C_n}{\sqrt{n}} e^{-2n\pi\varepsilon} = O\left(\frac{H}{\sqrt{\varepsilon}}\right).$$

Hence, from (2.762),

$$\int_0^{\infty} t^{-\frac{1}{2}} e^{-2\varepsilon t}(I(2t, 2H))^2\, dt = O\left(\frac{H}{\sqrt{\varepsilon}}\right) + O\left(f(H)\varepsilon^{-\frac{1}{6}}\right) = O\left(\frac{H}{\sqrt{\varepsilon}}\right),$$

$$\int_0^{\infty} e^{-2\varepsilon t}(I(2t, 2H))^2\, dt = \int_0^{\infty} e^{-\varepsilon t} t^{\frac{1}{2}} \cdot t^{-\frac{1}{2}} e^{-\varepsilon t} I^2\, dt = O\left(\frac{1}{\sqrt{\varepsilon}}\right)\int_0^{\infty} t^{-\frac{1}{2}} e^{-\varepsilon t} I^2\, dt = O\left(\frac{H}{\varepsilon}\right).$$

This is equivalent to (2.711); and the lemma follows.

Proof of Theorem A.

2.8. We defined I by

$$I = I(t, H) = \int_t^{t+H} X(u)\, du,$$

and we now define $\underset{\sim}{I}$ by

$$\underset{\sim}{I} = \underset{\sim}{I}(t, H) = \int_t^{t+H} |X(u)|\, du.$$

It is plain that $\underset{\sim}{I} = |I|$ if there is no zero of $X(u)$ in $(t, t+H)$.

If

$$\eta(s) = (1 - 2^{1-s})\zeta(s) = 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad (s = \tfrac{1}{2} + it),$$

we have

$$|X(t)| > A\left|\frac{\eta(s)}{1 - 2^{1-s}}\right| > A|\eta(s)|.$$

Hence

(2.81) $$\underset{\sim}{I} > A\int_t^{t+H} |\eta(s)|\, dt > A\,\Re\int_t^{t+H} \eta(s)\, dt$$

$$= AH + A\,\Re\left(i\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^{\frac{1}{2}+i(t+H)}\log n} - i\sum_{n+2}^{\infty} \frac{(-1)^{n-1}}{n^{\frac{1}{2}+it}\log n}\right)$$

$$= AH + \Psi,$$

say, where

$$\Psi = O(|\psi(t+H)| + |\psi(t)|),$$

in the notation of 2.4.

We denote by $\underset{\sim}{T}$ the interval $(T, 2T)$ and by $\underset{\sim}{U}$ the subset of $\underset{\sim}{T}$ in which

$$\underset{\sim}{I} < \tfrac{1}{2}AH, \tag{2.82}$$

A being the *same* constant as occurs in (2.81). Then $|\Psi| > \frac{1}{2}AH$ in $\underset{\sim}{U}$. But, by Lemma 7,

$$\int_T^{2T} |\Psi|^2\, dt < BT,$$

B, like A, being an absolute constant. Hence, if $m\underset{\sim}{U}$ is the measure of $\underset{\sim}{U}$, we have

$$\tfrac{1}{4}A^2H^2 m\underset{\sim}{U} < BT,$$

$$m\underset{\sim}{U} < \varepsilon_H T, \tag{2.83}$$

where ε_H is a number which tends to zero when $H \to \infty$. Thus

$$\underset{\sim}{I} > \tfrac{1}{2}AH \tag{2.84}$$

if t lies in $\underset{\sim}{T}$ and H is sufficiently large, except perhaps in a subset $\underset{\sim}{U}$ of $\underset{\sim}{T}$ whose measure is less than $\varepsilon_H T$.

On the other hand, by Lemma 11, we have

$$\int_T^{2T} I^2\, dt < CHT.$$

If then

$$|I| > \tfrac{1}{2}AH \tag{2.85}$$

in a subset $\underset{\sim}{V}$ of $\underset{\sim}{T}$, of measure $m\underset{\sim}{V}$, we have

$$\tfrac{1}{4}A^2H^2 m\underset{\sim}{V} < CHT,$$

$$m\underset{\sim}{V} < \varepsilon_H T. \tag{2.86}$$

Comparing (2.82), (2.85), (2.83) and (2.86), we see that

$$|I| < \underset{\sim}{I} \tag{2.87}$$

throughout all $\underset{\sim}{T}$ except a subset $\underset{\sim}{S}$ of measure less than $\varepsilon_H T$.

2.9. Divide $\underset{\sim}{T}$ into $\left[\frac{T}{2H}\right]$ pairs of abutting intervals j_1, j_2, each, except the last j_2, of length H, and each j_2 lying immediately to the right of the corresponding j_1. Then *either j_1 or j_2 contains a zero of $X(t)$, unless j_1 consists entirely of points of $\underset{\sim}{S}$.* If the second case occurs for ν j_1's, we have $\nu H < \varepsilon_H T$ or

$$\nu < \frac{\varepsilon_H}{H} T.$$

And therefore there are, in T, at least

$$\left(\frac{1}{2} - \varepsilon_H\right)\frac{T}{H} > \frac{T}{4H}$$

zeros, if H is sufficiently large; which proves the theorem.

3. The approximate functional equation.

3.1. Lemma 12. *Suppose that* σ *is fixed,* $0 < \sigma < 1$, $t > A$, $\xi > A$, *and*

(3.11) $$I = I(\xi, s) = \int_\xi^\infty u^{-s} {\cos \atop \sin} u\, du.$$

Then

(3.12) $$I = \Gamma(1-s) {\sin \atop \cos} \frac{1}{2} s\pi + O\left(\frac{\xi^{1-\sigma}}{t}\right) \qquad (\xi < At < t),$$

(3.13) $$I = \Gamma(1-s) {\sin \atop \cos} \frac{1}{2} s\pi + O\left(\frac{\xi^{2-\sigma}}{t(t-\xi)}\right) \qquad (At < \xi < t),$$

(3.14) $$I = O\left(\frac{\xi^{1-\sigma}}{\xi - t}\right) \qquad (t < \xi < At),$$

(3.15) $$I = O(\xi^{-\sigma}) \qquad (t < At < \xi),$$

and

(3.16) $$I = O(\xi^{-\sigma}\sqrt{t})$$

in any case.

It is sufficient to consider the integral which contains $\cos u$; and we suppose first that $\xi > t$. We have

$$2I = \int_\xi^\infty u^{-s} e^{iu}\, du + \int_\xi^\infty u^{-s} e^{-iu}\, du = I' + I'',$$

say. In the first place

$$I'' = \int_\xi^\infty u^{-\sigma} e^{-i(u + t\log u)}\, du = \int_\xi^\infty u^{-\sigma} e^{iw}\, du,$$

where

$$w = u + t \log u.$$

Since u and w increase together, we have

$$I'' = \int_{u=\xi}^{u=\infty} u^{-\sigma} \frac{e^{-iw}}{1 + \frac{t}{u}}\, dw.$$

The real part of I'' is

$$\int_{u=\xi}^{u=\infty} u^{-\sigma}\frac{\cos w}{1+\frac{t}{u}}\,dw = \xi^{-\sigma}\int_{u=\xi}^{u=\xi'}\frac{\cos w}{1+\frac{t}{u}}\,dw \qquad (\xi < \xi')$$

$$= \frac{\xi^{-\sigma}}{1+\frac{t}{\xi'}}\int_{u=\xi''}^{u=\xi'}\cos w\,dw \qquad (\xi < \xi'' < \xi')$$

$$= O(\xi^{-\sigma}).$$

The imaginary part may be treated in the same way. Hence, in proving (3.14) and (3.15), we need only consider I'.

Again

$$I' = \int_\xi^\infty u^{-\sigma}e^{i(u-t\log u)}\,du = \int_\xi^\infty u^{-\sigma}e^{iw}\,du,$$

where

$$w = u - t\log u.$$

Since $\xi > t$,

$$\frac{dw}{du} = 1 - \frac{t}{u} > 0,$$

so that u and w increase together. Hence

$$I' = \int_{u=\xi}^{u=\infty} u^{-\sigma}\frac{e^{iw}}{1-\frac{t}{u}}\,dw.$$

The real part of I' is

$$\int_{u=\xi}^{u=\infty} u^{-\sigma}\frac{\cos w}{1-\frac{t}{u}}\,dw = \frac{\xi^{-\sigma}}{1-\frac{t}{\xi}}\int_{u=\xi}^{u=\xi'}\cos w\,dw = O\left(\frac{\xi^{1-\sigma}}{\xi-t}\right) \qquad (\xi < \xi');$$

and similarly for the imaginary part. Since

$$\frac{\xi}{\xi-t} = O(1) \qquad (\xi > At > t),$$

this proves (3.14) and (3.15).

Next, suppose $\xi < t$. Then

$$I = \Gamma(1-s)\sin\tfrac{1}{2}s\pi - \int_0^\xi u^{-s}\cos u\,du$$

$$= \Gamma(1-s)\sin\tfrac{1}{2}s\pi - \frac{\xi^{1-s}}{1-s}\cos\xi - \frac{1}{1-s}\int_0^\xi u^{1-s}\sin u\,du$$

$$= \Gamma(1-s)\sin\tfrac{1}{2}s\pi + O\left(\frac{\xi^{1-\sigma}}{t}\right) - \frac{1}{1-s}\int_0^\xi u^{1-s}\sin u\,du.$$

Hence, in order to prove (3. 12) and (3. 13), it is enough to prove that

$$I_1 = \int_0^{\xi} u^{1-s} \sin u \, du = O\left(\frac{\xi^{2-\sigma}}{t-\xi}\right). \tag{3. 17}$$

Now

$$2 i I_1 = \int_0^{\xi} u^{1-s} e^{iu} du - \int_0^{\xi} u^{1-s} e^{-iu} du = I_1' - I_1'',$$

say. In the first place

$$I_1'' = \int_0^{\xi} u^{1-\sigma} e^{-i(u+t\log u)} du = \int_0^{\xi} u^{1-\sigma} e^{-iw} du,$$

where $w = u + t \log u$. The real part of I_1'' is

$$\int_{u=0}^{u=\xi} u^{1-\sigma} \frac{\cos w}{1+\frac{t}{u}} dw = \frac{\xi^{1-\sigma}}{1+\frac{t}{\xi}} \int_{u=\xi'}^{u=\xi} \cos w \, dw = O(\xi^{1-\sigma}) \qquad (0 < \xi' < \xi);$$

and similarly for the imaginary part. Hence, in proving (3. 17), we may confine our attention to I_1'.

Now

$$I_1' = \int_0^{\xi} u^{1-\sigma} e^{i(u-t\log u)} du = \int_0^{\xi} u^{1-\sigma} e^{iw} du,$$

where $w = u - t \log u$. As $\xi < t$,

$$\frac{dw}{du} = 1 - \frac{t}{u} < 0,$$

so that w decreases as u increases. The real part of I_1' is

$$\int_{u=0}^{u=\xi} u^{1-\sigma} \frac{\cos w}{1-\frac{t}{u}} dw = -\frac{\xi^{1-\sigma}}{\frac{t}{\xi}-1} \int_{u=\xi'}^{u=\xi} \cos w \, dw = O\left(\frac{\xi^{2-\sigma}}{t-\xi}\right) \qquad (0 < \xi' < \xi).$$

The imaginary part may be treated in the same way, so that we obtain (3. 17), and therefore (3. 12) and (3. 13).

It remains to prove (3. 16). If $\xi \geqq t + \sqrt{t}$, we have

$$I = O\left(\frac{\xi^{1-\sigma}}{\xi - t}\right) = O\left(\xi^{-\sigma} \frac{\xi}{\xi - t}\right) = O(\xi^{-\sigma} \sqrt{t});$$

and if $\xi \leqq t - \sqrt{t}$ we have

$$I = \Gamma(1-s) \sin \tfrac{1}{2} s \pi + O\left(\frac{\xi^{2-\sigma}}{t(t-\xi)}\right) = O\left(t^{\frac{1}{2}-\sigma}\right) + O\left(\xi^{-\sigma} \frac{\xi}{t-\xi}\right) = O(\xi^{-\sigma} \sqrt{t}).$$

We may suppose then that $t-\sqrt{t}<\xi \leqq t+\sqrt{t}$. If $\xi<t$, we write

$$I=\int_{\xi}^{t}+\int_{t}^{t+\sqrt{t}}+\int_{t+\sqrt{t}}^{\infty},$$

and if $\xi > t$ we write

$$I=-\int_{t}^{\xi}+\int_{t}^{t+\sqrt{t}}+\int_{t+\sqrt{t}}^{\infty};$$

and it is plainly enough to show that

$$(3.18)\qquad \int_{t}^{\eta} u^{-s}e^{iu}\,du = O\left(t^{\frac{1}{2}-\sigma}\right) \qquad (t-\sqrt{t}\leqq \eta \leqq t+\sqrt{t});$$

and this is obvious, since the integrand is $O(t^{-\sigma})$.

Hence we obtain (3.18), and the proof of Lemma 12 is completed.

The lemma was stated for positive values of t. The corresponding results when t is negative may be written down at once, by appropriate changes of t into $-t$.

Lemma 13. *The equations* (3.14) *and* (3.15) *of Lemma* 12 *hold for any positive value of* σ.

In fact, in proving these equations, no use was made of the assumption that $\sigma > 1$.

3.2. Lemma 14. *If* σ *is fixed and*

$$0<\sigma<1,\quad x>A,\quad y>A,\quad 2\pi xy=|t|,$$

then

$$\zeta(s)=\sum_{n<x} n^{-s}+\chi\sum_{n<y} n^{s-1}+O(x^{-\sigma})+O\left(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}\right),$$

where

$$\chi=\left(\frac{|t|}{2\pi e}\right)^{-it}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma}e^{\frac{1}{4}\pi i\,\mathrm{sgn}\,t}.$$

This lemma (the 'approximate functional equation') is important in various parts of the theory of $\zeta(s)$. At present, however, we shall be content to prove it in an imperfect form, which follows more naturally from our previous analysis and is sufficient for our immediate purpose. We accordingly reserve the proof of Lemma 14 for publication elsewhere, and here prove only

Lemma 15. *Under the conditions of Lemma* 14,

$$\zeta(s)=\sum_{n<x} n^{-s}+\chi\sum_{n<y} n^{s-1}+O\left(\left(x^{-\sigma}+|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}\right)\log|t|\right).$$

We may plainly suppose $t > 0$. We have, by Lemma 1,

$$(3.21)\quad \zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n \leq x} n^{-s} = 2(2\pi)^{s-1} \sum_{n=1}^{\infty} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s} \cos u\, du$$

$$= 2(2\pi)^{s-1}\left(\sum_{n<By} + \sum_{By<n\leq Cy} + \sum_{Cy<n}\right) = S_1 + S_2 + S_3,$$

say, B and C being constants and $0 < B < 1 < C$.

We begin by considering S_3. We have

$$2(2\pi)^{s-1} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s} \cos u\, du = -\frac{x^{-s}}{n\pi} \sin 2n\pi x + 2s(2\pi)^{s-1} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s-1} \sin u\, du,$$

$$S_3 = -\frac{x^{-s}}{\pi} \sum_{Cy<n} \frac{\sin 2n\pi x}{n} + 2s(2\pi)^{s-1} \sum_{Cy<n} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s-1} \sin u\, du.$$

The first term here is $O(x^{-\sigma})$, since $\sum \frac{\sin 2n\pi x}{n}$ is boundedly convergent. The second is

$$O\left(t \sum_{Cy<n} n^{\sigma-1} (nx)^{-\sigma-1}\right) = O\left(\frac{tx^{-\sigma-1}}{y}\right) = O(x^{-\sigma}),$$

by (3.15) and Lemma 13. Hence

$$(3.22)\qquad S_3 = O(x^{-\sigma}).$$

Next, we have, by (3.12),

$$(3.23)\quad S_1 = 2(2\pi)^{s-1} \Gamma(1-s) \sin \tfrac{1}{2} s\pi \sum_{n<By} n^{s-1} + S_1'' = S_1' + S_1'',$$

where

$$(3.24)\quad S_1'' = O\left(\sum_{n<By} n^{\sigma-1} \frac{(nx)^{1-\sigma}}{t}\right) = O\left(\frac{yx^{1-\sigma}}{t}\right) = O(x^{-\sigma}).$$

From (3.23) and (3.24) we deduce

$$(3.25)\qquad S_1 + S_3 = S_1' + O(x^{-\sigma}).$$

It remains to discuss S_2.

3.3. We have

$$(3.31)\qquad S_2 = 2(2\pi)^{s-1} \sum_{By\leq n\leq Cy} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s} \cos u\, du$$

$$= 2(2\pi)^{s-1}\left(\sum_{By\leq n < y-A} + \sum_{y-A\leq n\leq y+A} + \sum_{y+A<n\leq Cy}\right)$$

$$= S_2' + S_2'' + S_2''',$$

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say. By (3.16),

(3.321) $$S_2'' = O\Big(\sum_{y-A<n<y+A} n^{\sigma-1}(nx)^{-\sigma}\sqrt{t}\Big) = O\Big(\frac{x^{-\sigma}}{y}\sqrt{t}\Big) = O\big(t^{\frac12-\sigma}y^{\sigma-1}\big).$$

Secondly, by (3.14),

(3.322) $$S_2''' = O\Big(\sum_{y+A<n<Cy} n^{\sigma-1}\frac{(nx)^{1-\sigma}}{2n\pi x - t}\Big)$$
$$= O\Big(x^{-\sigma}\sum_{y+A<n<Cy}\frac{1}{n-y}\Big) = O(x^{-\sigma}\log y) = O(x^{-\sigma}\log t).$$

Finally, by (3.13)

(3.323) $$S_2' = 2(2\pi)^{s-1}\Gamma(1-s)\sin\tfrac12 s\pi \sum_{By\leq n<y-A} n^{s-1} + S'_{2,3} - S'_{2,1} + S'_{2,2},$$

where

(3.324) $$S'_{2,2} = O\Big(\sum_{By\leq n<y-A} n^{\sigma-1}\frac{(nx)^{2-\sigma}}{t(t-2n\pi x)}\Big)$$
$$= O\Big(\frac{x^{1-\sigma}y}{t}\sum_{By\leq n<y-A}\frac{1}{y-n}\Big) = O\Big(\frac{x^{1-\sigma}y\log t}{t}\Big) = O(x^{-\sigma}\log t).$$

From (3.21), (3.25), (3.31), (3.321), (3.322), (3.323) and (3.324), we deduce

(3.33) $$\zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n<x} n^{-s} - S_1' + S'_{2,1} + O(x^{-\sigma}\log t) + O(t^{\frac12-\sigma}y^{\sigma-1})$$
$$= 2(2\pi)^{s-1}\Gamma(1-s)\sin\tfrac12 s\pi\sum_{n<y-A} n^{s-1} + O(x^{-\sigma}\log t) + O(t^{\frac12-\sigma}y^{\sigma-1}).$$

Now

$$\frac{x^{1-s}}{1-s} = O\Big(\frac{x^{1-\sigma}}{t}\Big) = O(x^{-\sigma}),\ 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac12 s\pi = \chi\Big(1+O\Big(\frac1t\Big)\Big),$$
$$\chi O\Big(\frac1t\Big)\sum_{n<y-A} n^{s-1} = O(t^{-\frac12-\sigma}y^{\sigma}) = O(t^{\frac12-\sigma}y^{\sigma-1}),$$

and we may plainly replace $n\leqq x$ and $n<y-A$ by $n<x$ and $n<y$. Hence we obtain the result of Lemma 15.

4. Discussion of $\int_T^{T+U} I^2\,dt$.

Lemma 16. *If* $A<T<T'$, $0<\xi<T$, $0\leqq\gamma<T$, *then*

(4.11) $$\int_T^{T'}\Big(\frac{t}{e\xi}\Big)^{i(t+\gamma)}dt = O\Bigg(\frac{1}{\log\frac{T}{\xi}}\Bigg).$$

The integral is

$$j = \int_T^{T'} e^{iw}\,dt,$$

where

$$w = (t+\gamma)(\log t - 1 - \log \xi),$$

$$\frac{dw}{dt} = \log\frac{t}{\xi} + \frac{\gamma}{t} > 0,$$

$$\frac{d^2 w}{dt^2} = \frac{1}{t} - \frac{\gamma}{t^2} > 0.$$

The real part of j is

$$\int_{u=T}^{u=T'} \frac{\cos w}{\log\frac{t}{\xi} + \frac{\gamma}{t}}\,dw = \frac{1}{\log\frac{T}{\xi} + \frac{\gamma}{T}} \int_{u=T}^{u=T''} \cos w\,dw = O\left(\frac{1}{\log\frac{T}{\xi}}\right) \quad (T < T'' < T'),$$

and similarly for the imaginary part.

Lemma 17. *If t is positive*

$$X(t) = \Theta + \bar{\Theta} + O(t^{-\frac{1}{4}}\log t),$$

where

$$\Theta = -\left(\frac{\pi}{2}\right)^{\frac{1}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i} \sum_{n<\tau} n^{-\frac{1}{2}-it}, \quad \tau = \sqrt{\frac{t}{2\pi}},$$

and $\bar{\Theta}$ is the conjugate of Θ.

Taking $\sigma = \frac{1}{2}$, $x = y = \tau$ in Lemma 15, we obtain

$$(4.12) \quad \zeta\left(\frac{1}{2} + it\right) = \sum_{n<\tau} n^{-\frac{1}{2}-it} + \left(\frac{t}{2\pi e}\right)^{-it} e^{\frac{1}{4}\pi i} \sum_{n<\tau} n^{-\frac{1}{2}+it} + O(t^{-\frac{1}{4}}\log t).$$

But, by (2.61),

$$\zeta\left(\frac{1}{2} + it\right) = -\left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{\frac{1}{8}\pi i}\left(\frac{t}{2\pi e}\right)^{-\frac{1}{2}it} X(t)\left(1 + O\left(\frac{1}{t}\right)\right).$$

Substituting in (4.12), and observing that

$$O\left(\frac{1}{t}\right)\sum_{n<\tau} n^{-\frac{1}{2}} = O(t^{-\frac{3}{4}}) = O(t^{-\frac{1}{4}}\log t),$$

we obtain the result of the lemma.

4.2. We suppose now that

$$(4.21) \qquad \tfrac{1}{2} < a < b$$

and, for the present, that

$$(4.22) \qquad b = \tfrac{5}{8};$$

that

(4.23) $$T^a < U < T^b$$

and

(4.24) $$0 < H < T^c,$$

where $c = c(a)$ is a positive constant which will be chosen small enough to satisfy a number of conditions appearing in the sequel; and that

(4.25) $$J = J(T, U) = \int_T^{T+U} I^2\,dt,$$

where

(4.26) $$I = I(t, H) = \int_t^{t+H} X(u)\,du.$$

From this point onwards A, B, C, ... and the constants of the O's depend upon a only.

We shall now prove

Lemma 18. *If $T^a < U < T^b$, where $a > \frac{1}{2}$, and $0 < H < T^c$, where c is positive and sufficiently small, then*

$$J = \int_T^{T+U} I^2\,dt = \pi\sqrt{2\pi}\,HU + O\left(\frac{U}{\log T}\right).$$

Suppose that $0 \leqq \alpha \leqq H$, $0 \leqq \beta \leqq H$. Then

$$X(t+\alpha) = \Theta_\alpha + \overline{\Theta}_\alpha + O(t^{-\frac{1}{4}}\log t),$$

where Θ_α is obtained from Θ by writing $t+\alpha$ in the place of t. Since $\sqrt{t+\alpha} - \sqrt{t} = o(1)$, we may replace the limits of summation in Θ_α by $n < \tau$. Also

$$\left(\frac{t+\alpha}{2\pi e}\right)^{\frac{1}{2}i(t+\alpha)} = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}i(t+\alpha)} e^{\frac{1}{2}i\alpha}\left(1 + O\left(\frac{\alpha^2}{t}\right)\right),$$

and the contribution of the error term here to Θ_α is

$$O\left(\frac{\alpha^2}{t}t^{\frac{1}{4}}\right) = O(t^{-\frac{1}{4}}),$$

if c is small enough. Hence

(4.27) $$X(t+\alpha) = \Phi_\alpha + \overline{\Phi}_\alpha + O(t^{-\frac{1}{4}}\log t),$$

where

(4.28) $$\Phi_\alpha = -\left(\frac{\pi}{2}\right)^{\frac{1}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}i(t+\alpha)} e^{\frac{1}{2}i\alpha - \frac{1}{8}\pi i}\sum_{n<\tau} n^{-\frac{1}{2}-i(t+\alpha)};$$

and

$$(4.29)\quad X(t+\alpha)\,X(t+\beta) = \Phi_\alpha \Phi_\beta + \bar\Phi_\alpha \bar\Phi_\beta + \Phi_\alpha \bar\Phi_\beta + \bar\Phi_\alpha \Phi_\beta + O\Big(t^{-\frac14}\log t \Big|\sum_{n<\tau} n^{-\frac12 - i(t+\alpha)}\Big|\Big) + O\Big(t^{-\frac14}\log t \Big|\sum_{n<\tau} n^{-\frac12 - i(t+\beta)}\Big|\Big) + O(t^{-\frac12}(\log t)^2) = P + \bar P + Q + \bar Q + R_1 + R_2 + O(t^{-\frac12}(\log t)^2),$$

say.

4.3. We shall prove first that

$$(4.31)\qquad \int_T^{T+U} R_1\,dt = O(UT^{-A}), \qquad \int_T^{T+U} R_2\,dt = O(UT^{-A}),$$

uniformly for $0 \leqq \alpha \leqq H$, $0 \leqq \beta \leqq H$. It is sufficient to consider the first integral.

If $\underline{T} = \sqrt{\frac{T}{2\pi}}$, we have

$$(4.32)\quad \sum_{n<\tau} n^{-\frac12 - i(t+\alpha)} - \sum_{n<\underline{T}} n^{-\frac12 - i(t+\alpha)} = O\left(\frac{U}{\sqrt{T}}\,\frac{1}{T^{\frac14}}\right) = O(T^{-A}),$$

by (4.22) and (4.23), and so

$$R_1 = O\Big(t^{-\frac14}\log t \Big|\sum_{n<\underline{T}} n^{-\frac12 - i(t+\alpha)}\Big|\Big) + O(T^{-A}).$$

Hence

$$(4.33)\quad \int_T^{T+U} R_1\,dt = O\left(\int_T^{T+U} t^{-\frac14}\log t \Big|\sum_{n<\underline{T}} n^{-\frac12 - i(t+\alpha)}\Big|\,dt\right) + O(UT^{-A}).$$

The first term on the right hand side is

$$O\left(T^{-A}\int_T^{T+U}\Big|\sum_{n<\underline{T}}\Big|\,dt\right) = O\left(T^{-A}\sqrt{U}\left(\int_T^{T+U}\Big|\sum_{n<\underline{T}}\Big|^2 dt\right)^{\frac12}\right).$$

But

$$\int_T^{T+U}\Big|\sum_{n<\underline{T}}\Big|^2 dt = \sum_{m,n<\underline{T}} \frac{1}{\sqrt{mn}} \int_T^{T+U} \left(\frac{m}{n}\right)^{i(t+\alpha)} dt = U\sum_{n<\underline{T}}\frac{1}{n} + O\left(\sum_{m,n<\underline{T}}' \frac{1}{\sqrt{mn}\,\left|\log\frac{m}{n}\right|}\right)^{9)} = O(U\log T) + O(\underline{T}\log T) = O(U\log T),$$

by Lemma 5, (4.21), and (4.23). Hence

$$(4.34)\quad T^{-A}\int_T^{T+U}\Big|\sum_{n<\underline{T}}\Big|\,dt = O\left(T^{-A}\sqrt{U}\sqrt{U\log T}\right) = O(UT^{-A}),$$

and (4.31) now follows from (4.33) and (4.34).

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[9]) The dash implies as usual that $m \neq n$.

4.4. Next we prove that

$$(4.41) \qquad \int_T^{T+U} P\,dt = O(UT^{-A}), \qquad \int_T^{T+U} \overline{P}\,dt = O(UT^{-A}).$$

It is sufficient to consider the first integral. We begin by replacing the limits of summation in Φ_α and Φ_β by limits of the type $n < T$. By (4.23),

$$\Big(\sum_{m,n<\tau} - \sum_{m,n<T}\Big) m^{-\frac{1}{2}-i(t+\alpha)}\, n^{-\frac{1}{2}-i(t+\beta)}$$

$$= O\Big(T^{-A}\Big|\sum_{m<T} m^{-\frac{1}{2}-i(t+\alpha)}\Big|\Big) + O\Big(T^{-A}\Big|\sum_{n<T} n^{-\frac{1}{2}-i(t+\beta)}\Big|\Big) + O(T^{-A}).$$

The last term obviously gives rise to an error $O(UT^{-A})$, and the argument of the last section shows that the same is true of the first two terms. Hence the change of the limits of summation is irrelevant to the argument.

Now

$$(4.42) \qquad \int_T^{T+U} \Big(\frac{t}{2\pi e}\Big)^{it+\frac{1}{2}i(\alpha+\beta)} \Big(\sum_{m,n<T} m^{-\frac{1}{2}-i\alpha}\, n^{-\frac{1}{2}-i\beta}\,(mn)^{-it}\Big)\,dt$$

$$= O\Big(\sum_{m,n<T} \frac{1}{\sqrt{mn}}\Big|\int_T^{T+U} \Big(\frac{t}{2\pi e mn}\Big)^{it+\frac{1}{2}i(\alpha+\beta)} dt\Big|\Big).$$

If we write $2\pi mn = \xi$ and $\frac{1}{2}(\alpha+\beta) = \gamma$, we have $\xi \leqq T$ and $0 \leqq \gamma \leqq H < T$. Hence we may apply Lemma 16, and the integral on the right hand side of (4.42) is

$$O\left(\frac{1}{\log \frac{T}{2\pi mn}}\right).$$

If $m \neq n$ we have

$$\log\frac{T}{2\pi mn} > \log\Big(\frac{\operatorname{Max}(m^2, n^2)}{mn}\Big) = \Big|\log\frac{m}{n}\Big|;$$

and if $m = n < T - 1$ we have

$$\log\frac{T}{2\pi mn} = \log\frac{T}{2\pi m^2} = 2\log\frac{T}{m}.$$

These results give us upper bounds for all the terms in (4.42) except that for which m and n have each their greatest value, and this term is plainly $O\left(\frac{U}{T}\right)$. We thus obtain

$$\int_T^{T+U} = O\left(\sum_{m,n<T}{}' \frac{1}{\sqrt{mn}\left|\log \frac{m}{n}\right|}\right) + O\left(\sum_{m<T-1} \frac{1}{m \log \frac{T}{m}}\right) + O\left(\frac{U}{T}\right)$$

$$= O(T \log T) + O\left(\sum_{m<\frac{1}{2}\underline{T}} \frac{1}{m}\right) + O\left(\sum_{\frac{1}{2}T \leq m < T-1} \frac{T}{m}\right) + O\left(\frac{U}{T}\right)$$

$$= O(T \log \underline{T}) + O(\log \underline{T}) + O(T \log T) + O\left(\frac{U}{T}\right) = O(UT^{-A}),$$

since $U > T^a$ and $\underline{T} = O(\sqrt{T})$. We have thus proved (4.41).

4.5. From (4.29), (4.31), and (4.41) it follows that

$$(4.51) \quad \int_T^{T+U} X(t+\alpha)\, X(t+\beta)\, dt = \int_T^{T+U} (Q + \bar{Q})\, dt + O(UT^{-A}).$$

It is clear moreover that, in Q and $\bar{Q}$, we may modify the limits of summation in the same way as in P and $\bar{P}$. Denoting the modified forms of Q and $\bar{Q}$ by Q_1 and $\bar{Q}_1$, we have

$$(4.52) \quad J = \int_T^{T+U} I^2\, dt = \int_T^{T+U} dt \int_0^H \int_0^H X(t+\alpha)\, X(t+\beta)\, d\alpha\, d\beta$$

$$= \int_0^H \int_0^H d\alpha\, d\beta \int_T^{T+U} X(t+\alpha)\, X(t+\beta)\, d\alpha\, d\beta$$

$$= \int_0^H \int_0^H d\alpha\, d\beta \int_T^{T+U} (Q_1 + \bar{Q}_1)\, dt + O(H^2 U T^{-A})$$

$$= \underset{\sim}{J} + O(H^2 U T^{-A}) = \underset{\sim}{J} + O(U T^{-A}),$$

say, if c is sufficiently small.

4.6. The value of $\underset{\sim}{J}$ is, by (4.28) and (4.29),

$$(4.61) \quad \underset{\sim}{J} = \sqrt{2\pi}\, \Re \left[\int_0^H \int_0^H d\alpha\, d\beta \int_T^{T+U} \left(\frac{t}{2\pi}\right)^{\frac{1}{2}i(\alpha-\beta)} \sum_{m,n<\underline{T}} m^{-\frac{1}{2}-i\alpha}\, n^{-\frac{1}{2}+i\beta} \left(\frac{n}{m}\right)^{it} dt \right].$$

We begin the discussion of the right hand side by showing that the contribution of the terms for which $m \neq n$ is $O(UT^{-A})$.

If $m \neq n$, write $\frac{n}{m} = e^{\lambda}$ and $\frac{1}{2}(\alpha - \beta) = \gamma$, so that $|\gamma| < H$. Then

$$\int_T^{T+U} t^{i\gamma} e^{\lambda i t}\, dt = \left[\frac{t^{i\gamma} e^{\lambda i t}}{\lambda i}\right]_T^{T+U} - \frac{\gamma}{\lambda} \int_T^{T+U} t^{i\gamma - 1} e^{\lambda i t}\, dt = O\left(\frac{1}{|\lambda|}\right) + O\left(\frac{HU}{|\lambda| T}\right) = O\left(\frac{1}{|\lambda|}\right).$$

Hence the terms in which $m \neq n$ contribute, when we integrate with respect to t,

$$O\left(\sum_{m,n<\underline{T}}{}' \frac{1}{\sqrt{mn}\left|\log \frac{m}{n}\right|}\right) = O(\sqrt{T} \log T),$$

and when we integrate with respect to α and β,

$$O(H^2 \sqrt{T} \log T) = O(UT^{-A}),$$

if once more c is sufficiently small.

We have therefore, from (4.52) and (4.61),

$$(4.62) \quad J = \sqrt{2\pi}\, \Re \left[\sum_{n<T} \frac{1}{n} \int_0^H \int_0^H d\alpha\, d\beta \int_T^{T+U} \left(\frac{t}{2\pi n^2}\right)^{\frac{1}{2}i(\alpha-\beta)} dt \right] + O(UT^{-A}).$$

4.7. Write

$$\frac{t}{2\pi n^2} = v_n = v.$$

Then

$$\int_0^H \int_0^H v^{\frac{1}{2}i(\alpha-\beta)}\, d\alpha\, d\beta = \left(\frac{\sin\left(\frac{1}{4} H \log v\right)}{\frac{1}{4}\log v}\right)^2 = V_n,$$

say. Also

$$V_n = 8\, \frac{1-\cos\left(\frac{1}{2} H \log v\right)}{(\log v)^2} = 2\int_0^H d H_1 \int_0^{H_1} \cos\left(\frac{1}{2} H_2 \log v\right) d H_2.$$

Hence, if $|u-n| \leqq 1$,

$$V_u - V_n = 2\int_0^H d H_1 \int_0^{H_1} \left(\cos\left(\frac{1}{2} H_2 \log \frac{t}{2\pi u^2}\right) - \cos\left(\frac{1}{2} H_2 \log \frac{t}{2\pi n^2}\right)\right) d H_2.$$

The difference of cosines here is

$$O(1)\, O\left(\frac{H}{n}\right) = O\left(\frac{H}{n}\right),$$

and

$$(4.71) \qquad V_u - V_n = O\left(\frac{H^3}{n}\right).$$

We have now, from (4.62),

$$J + O(UT^{-A}) = \sqrt{2\pi} \sum_{n<T} \frac{1}{n} \int_T^{T+U} V_n\, dt = \left(\sqrt{2\pi} \sum_{n<T^{\frac{1}{4}}} + \sqrt{2\pi} \sum_{T^{\frac{1}{4}} \leqq n < T}\right) = J_1 + J_2,$$

say. The first sum is

$$O\left(\sum_{n<T^{\frac{1}{4}}} \frac{1}{n} \int_T^{T+U} \frac{dt}{(\log T)^2}\right) = O\left(\frac{U}{\log T}\right).$$

In J_2 we may replace summation with respect to n by integration with respect to u, with an error

$$O\left(\sum_{T^{\frac{1}{4}} \leqq n < T} \frac{1}{n} \cdot U \cdot \frac{H^3}{n}\right) = O(H^3 U T^{-A}) = O(U T^{-A}),$$

if c is small enough. Thus

$$(4.72)\qquad J=\sqrt{2\pi}\int_T^{T+U}dt\int_{T^{\frac{1}{4}}}^{\underline{T}}\left(\frac{\sin\left(\frac{1}{4}H\log\frac{t}{2\pi u^2}\right)}{\frac{1}{4}\log\frac{t}{2\pi u^2}}\right)^2\frac{du}{u}+O\left(\frac{U}{\log T}\right).$$

4.8. In (4.72) we may replace $\underline{T}=\sqrt{\frac{T}{2\pi}}$ by $\tau=\sqrt{\frac{t}{2\pi}}$. For since $\tau-\underline{T}=O\left(\frac{U}{\sqrt{T}}\right)$, the error thus introduced is

$$O\left(\int_T^{T+U}\frac{U}{\sqrt{T}}\cdot H^2\cdot\frac{1}{\sqrt{T}}dt\right)=O\left(\frac{H^2U^2}{T}\right)=O\left(\frac{U}{\log T}\right),$$

if c is small enough. Further, if we write

$$\frac{t}{2\pi u^2}=e^{\frac{4x}{H}},\quad -\frac{du}{u}=\frac{2\,dx}{H},$$

we obtain

$$\int_{T^{\frac{1}{4}}}^{\tau}\left(\frac{\sin\frac{1}{4}H\log\frac{t}{2\pi u^2}}{\frac{1}{4}\log\frac{t}{2\pi u^2}}\right)^2\frac{du}{u}=2H\int_0^{\xi}\left(\frac{\sin x}{x}\right)^2dx,$$

where

$$\xi=\frac{1}{4}H\log\frac{t}{2\pi\sqrt{T}}.$$

Thus the integral in (4.72) becomes

$$(4.73)\quad 2\sqrt{2\pi}\,H\int_T^{T+U}dt\int_0^{\xi}\left(\frac{\sin x}{x}\right)^2dx=2\sqrt{2\pi}\,H\int_T^{T+U}\left(\frac{1}{2}\pi+O\left(\frac{1}{\xi}\right)\right)dt$$
$$=\pi\sqrt{2\pi}\,H\,U+O\left(\frac{U}{\log T}\right).$$

Finally, from (4.72), and (4.73), we deduce

$$J=\pi\sqrt{2\pi}\,H\,U+O\left(\frac{U}{\log T}\right),$$

and Lemma 18 is proved, when $b=\frac{5}{8}$.

4.9. It is easy now to remove the restriction that $b=\frac{5}{8}$. Suppose only that $U>T^a$, and let

$$U_1=T^d,\quad d=\tfrac{1}{2}(a+\tfrac{1}{2})<a.$$

Then

$$U_1=O\left(\frac{U}{H\log T}\right),$$

if c is small enough, and

$$\int_{T+rU_1}^{T+(r+1)U_1} I^2\,dt = \pi\sqrt{2\pi}\,H\,U_1 + O\left(\frac{U_1}{\log(T+rU_1)}\right) = \pi\sqrt{2\pi}\,H\,U_1 + O\left(\frac{U_1}{\log T}\right),$$

for $r = 0, 1, 2, \ldots, \nu - 1 = \left[\frac{U}{U_1}\right] - 1$. Adding all these equations, we obtain

$$\int_{T}^{T+\nu U_1} I^2\,dt = \pi\sqrt{2\pi}\,H\,U_1\left[\frac{U}{U_1}\right] + O\left(\frac{U}{\log T}\right) = \pi\sqrt{2\pi}\,H\,U + O\left(\frac{U}{\log T}\right).$$

Also

$$\int_{T+\nu U_1}^{T+U} I^2\,dt = O(H\,U_1) + O\left(\frac{U_1}{\log T}\right) = O\left(\frac{U}{\log T}\right).$$

The sum of the last two equations gives the result required.

5. Proof that $N_0(T+U) - N_0(T) > KU$.

5.1. Lemma 19. *If U and H satisfy the conditions of Lemma 18, and*

$$M = M(t, H) = \int_t^{t+H} \zeta(\tfrac{1}{2} + iu)\,du - H,$$

then

$$N = N(T, U) = \int_T^{T+U} |M|^2\,dt = O(U).$$

The proof of this lemma is very similar to that of Lemma 18. As there, we suppose initially that $b = \frac{5}{8}$ and $t > 0$. We have, by Lemma 15,

$$\begin{aligned}\varphi(t+\alpha) &= \zeta(\tfrac{1}{2} + it + i\alpha) - 1 \\ &= \sum_{2 \leq m < \tau_\alpha} m^{-\frac{1}{2} - it - i\alpha} + \left(\frac{t+\alpha}{2\pi e}\right)^{-i(t+\alpha)} e^{\frac{1}{4}\pi i} \sum_{n < \tau_\alpha} n^{-\frac{1}{2} + it + i\alpha} + O(T^{-A})\end{aligned}$$

where $\tau_\alpha = \sqrt{\frac{t+\alpha}{2\pi}}$. As in 4.2, we may replace τ_α by τ, and $\left(\frac{t+\alpha}{2\pi e}\right)^{-i(t+\alpha)}$ by $\left(\frac{t}{2\pi e}\right)^{-i(t+\alpha)} e^{-i\alpha}$. Thus

$$\varphi(t+\alpha) = \sum_{2 \leq m < \tau} m^{-\frac{1}{2} - it - i\alpha} + \left(\frac{t}{2\pi e}\right)^{-i(t+\alpha)} e^{\frac{1}{4}\pi i - i\alpha} \sum_{n < \tau} n^{-\frac{1}{2} + it + i\alpha} + O(T^{-A}),$$

$$\bar{\varphi}(t+\beta) = \sum_{2 \leq m < \tau} m^{-\frac{1}{2} + it + i\beta} + \left(\frac{t}{2\pi e}\right)^{i(t+\beta)} e^{-\frac{1}{4}\pi i + i\beta} \sum_{n < \tau} n^{-\frac{1}{2} - it - i\beta} + O(T^{-A}).$$

We have therefore

$$\varphi(t+\alpha)\,\bar{\varphi}(t+\beta) = P + Q_1 + Q_2 + R + S_1 + S_2 + S_3 + S_4 + W,$$

where

$$P = \sum m_1^{-\frac{1}{2}-it-i\alpha} m_2^{-\frac{1}{2}+it+i\beta},$$

$$Q_1 = \left(\frac{t}{2\pi e}\right)^{i(t+\beta)} e^{-\frac{1}{4}\pi i+i\beta} \sum m^{-\frac{1}{2}-it-i\alpha} n^{-\frac{1}{2}-it-i\beta},$$

Q_2 is a sum of the same type as Q_1,

$$R = \left(\frac{t}{2\pi}\right)^{i(\beta-\alpha)} \sum n_1^{-\frac{1}{2}+it+i\alpha} n_2^{-\frac{1}{2}-it-i\beta},$$

$$S_1 = O\left(T^{-A}\left|\sum m^{-\frac{1}{2}-it-i\alpha}\right|\right),$$

S_2, S_3, S_4 are sums of the same type as S_1, and

$$W = O(T^{-A}).$$

In the summations every m runs over the range $2 \leqq m < \tau$ and every n over the range $1 \leqq n < \tau$.

We write

$$\int_T^{T+U} \varphi(t+\alpha)\bar{\varphi}(t+\beta)\,dt = \int_T^{T+U}(P+Q_1+Q_2+R+S_1+S_2+S_3+S_4+W)\,dt$$
$$= P^0 + Q_1^0 + Q_2^0 + R^0 + S_1^0 + S_2^0 + S_3^0 + S_4^0 + W^0.$$

Obviously

(5.11) $$W^0 = O(UT^{-A});$$

and

(5.12) $$S_1^0 + S_2^0 + S_3^0 + S_4^0 = O(UT^{-A})$$

in virtue of the argument of 4.3.

Next

$$Q_1^0 = \int_T^{T+U} Q_1\,dt = O(1) \sum \frac{1}{\sqrt{mn}}\, m^{i(\beta-\alpha)} \int_T^{T+U} \left(\frac{t}{2\pi emn}\right)^{i(t+\beta)} dt$$
$$= O\left(\sum \frac{1}{\sqrt{mn}} \left|\int_T^{T+U} \left(\frac{t}{2\pi emn}\right)^{i(t+\beta)} dt\right|\right);$$

and so, by the argument of 4.4,

(5.13) $$Q_1^0 + Q_2^0 = O(UT^{-A}).$$

Thus

(5.14) $$\int_T^{T+U} \varphi(t+\alpha)\bar{\varphi}(t+\beta)\,dt = P^0 + R^0 + O(UT^{-A}).$$

5.2. Again

$$(5.21)\qquad R^0 = \int_T^{T+U} R\,dt = \sum n_1^{-\frac{1}{2}+i\alpha} n_1^{-\frac{1}{2}-i\beta} \int_T^{T+U} \left(\frac{t}{2\pi}\right)^{i(\beta-\alpha)} \left(\frac{n_1}{n_2}\right)^{it} dt$$

$$= \sum_{n_1=n_2} + \sum_{n_1\neq n_2} = R_1^0 + R_2^0,$$

say. Now

$$\int_T^{T+U} \left(\frac{t}{2\pi}\right)^{i(\beta-\alpha)} \left(\frac{n_1}{n_2}\right)^{it} dt = O\left(\frac{1}{\left|\log \frac{n_1}{n_2}\right|}\right) \qquad (n_1 \neq n_2),$$

by the argument of 4.6; and so

$$(5.22)\qquad R_2^0 = O\left(\sum_{n_1\neq n_2} \frac{1}{\sqrt{n_1 n_2}\left|\log\frac{n_1}{n_2}\right|}\right) = O(T\log T) = O(UT^{-A}).$$

Similarly

$$(5.23)\qquad P^0 = \int_T^{T+U} P\,dt = \sum m_1^{-\frac{1}{2}-i\alpha} m_2^{-\frac{1}{2}+i\beta} \int_T^{T+U} \left(\frac{m_2}{m_1}\right)^{it} dt$$

$$= \sum_{m_1=m_2} + \sum_{m_1\neq m_2} = P_1^0 + P_2^0,$$

and

$$(5.24)\qquad P_2^0 = O\left(\sum_{m_1\neq m_2} \frac{1}{\sqrt{m_1 m_2}\left|\log\frac{m_1}{m_2}\right|}\right) = O(UT^{-A}).$$

From (5.14), (5.21), (5.22), (5.23) and (5.24), it follows that

$$(5.25)\qquad \int_T^{T+U} \varphi(t+\alpha)\bar{\varphi}(t+\beta)\,dt = P_1^0 + R_1^0 + O(UT^{-A}).$$

5.3. Hence

$$(5.31)\qquad N = \int_T^{T+U} |M|^2\,dt = \int_T^{T+U} M\bar{M}\,dt$$

$$= \int_T^{T+U} dt \int_0^H (\zeta(\tfrac{1}{2}+it+i\alpha)-1)\,d\alpha \int_0^H (\zeta(\tfrac{1}{2}-it-i\beta)-1)\,d\beta$$

$$= \int_T^{T+U} dt \int_0^H \varphi(t+\alpha)\,d\alpha \int_0^H \bar{\varphi}(t+\beta)\,d\beta = \int_0^H d\alpha \int_0^H d\beta \int_T^{T+U} \varphi(t+\alpha)\bar{\varphi}(t+\beta)\,dt$$

$$= \int_0^H\int_0^H (P_1^0 + R_1^0)\,d\alpha\,d\beta + O(H^2 UT^{-A})$$

$$= \int_0^H\int_0^H (P_1^0 + R_1^0)\,d\alpha\,d\beta + O(UT^{-A}),$$

if c is sufficiently small.

Now

$$R_1^0 = \sum n^{i(\alpha-\beta)-1} \int_T^{T+U} \left(\frac{t}{2\pi}\right)^{i(\beta-\alpha)} dt,$$

(5. 32)

$$\int_0^H \int_0^H R_1^0 \, d\alpha \, d\beta = \sum \frac{1}{n} \int_T^{T+U} dt \int_0^H \int_0^H \left(\frac{t}{2\pi n}\right)^{i(\beta-\alpha)} d\alpha \, d\beta$$

$$= 4 \sum \frac{1}{n} \int_T^{T+U} \left(\frac{\sin\left(\frac{1}{2} H \log \frac{t}{2\pi n}\right)}{\log \frac{t}{2\pi n}} \right)^2 dt$$

$$= O\left(\sum \frac{1}{n} \, \frac{U}{(\log T)^2} \right) = O\left(\frac{U}{\log T}\right).$$

Further

$$P_1^0 = \sum m^{i(\beta-\alpha)-1} \int_T^{T+U} dt = U \sum m^{i(\beta-\alpha)-1},$$

(5. 33)

$$\int_0^H \int_0^H P_1^0 d\alpha \, d\beta = 4\, U \sum \frac{1}{m} \left(\frac{\sin\left(\frac{1}{2} H \log m\right)}{\log m}\right)^2 = O(U).$$

Finally, from (5. 25), (5. 32) and (5. 33), we deduce

$$N = O(U)$$

the result of the lemma, when $b = \frac{5}{8}$. This restriction on b may now be removed just as in 4. 9.

5. 4. As in 2. 8 we write

$$\underset{\sim}{I} = \int_t^{t+H} |X(u)| \, du.$$

Lemma 20. *There exists an A such that*

$$\underset{\sim}{I} > \tfrac{1}{2} H$$

throughout the interval $\underset{\sim}{T} = (T, T+U)$, except in a set $\underset{\sim}{S}$ of measure less than $\frac{AU}{H^2}$.

Provided we choose $A \geqq 16$, we may suppose $H \geqq 4$. We have then, by (2. 61)

$$\underset{\sim}{I} = \left(\frac{\pi}{2}\right)^{\frac{1}{4}} \int_t^{t+H} \left|\zeta\left(\tfrac{1}{2} + iu\right)\right| \left(1 + O\left(\frac{1}{u}\right)\right) du$$

$$> \left(\frac{\pi}{2}\right)^{\frac{1}{4}} \int_t^{t+H} \left|\zeta\left(\tfrac{1}{2} + iu\right)\right| du + O\left(H T^{\frac{1}{8}} \frac{1}{T}\right) > \left(\frac{\pi}{2}\right)^{\frac{1}{4}} \left(\Re \int_t^{t+H} \zeta\left(\tfrac{1}{2} + iu\right) du - 1\right)$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{4}} (H - 1 + \Re M) > \left(\frac{3}{4} H - |M|\right).$$

Hence, in $\underset{\sim}{S}$,

$$|M| \geqq \tfrac{1}{4} H.$$

Since $\int_T^{T+U} |M|^2 dt < A U$, we must have

$$(\tfrac{1}{4} H)^2 m \underset{\sim}{S} < A U,$$

whence the lemma.

5.5. From Lemma 18 we have

$$\int_T^{T+U} I^2 dt < A H U \qquad (H \geqq 4). \tag{5.51}$$

This inequality is sufficient for the deduction of Theorem B. Let $\underset{\sim}{S}'$ be the sub-set of $\underset{\sim}{T}$ for which $|I| \geqq \tfrac{1}{2} H$. By (5.51),

$$m \underset{\sim}{S}' < \frac{A U}{H}.$$

Now $\underset{\sim}{I} > |I|$, except possibly in $\underset{\sim}{S} + \underset{\sim}{S}'$. The measure of $\underset{\sim}{S} + \underset{\sim}{S}'$ is less than $\varepsilon_H U$, where ε_H is a function of H only which tends to zero as $H \to \infty$; and Theorem B follows by the argument used in 2.9 to establish Theorem A.

6. Remarks on the proof of Theorem B.

6.1. As was observed in 5.5, we do not use the full force of Lemma 18. The complete lemma, however, seems of considerable interest in itself, and it may prove to be of service in the future. At the moment, however, we are unable to derive from it any suggestion for a method for reducing the factor $\log T$ by which Theorem B falls short of what is doubtless the real truth. It is instructive to examine how our proof fails to give more, and we add in conclusion some remarks on this and on related points.

6.2. The inequality (5.51) may be replaced by the more precise relation

$$\frac{1}{U} \int_T^{T+U} \left(\frac{I}{H}\right)^2 dt = O\left(\frac{1}{H}\right) + O\left(\frac{1}{H^2 \log T}\right). \tag{6.21}$$

Now $I : H$ is the mean of $X(u)$ in t to $t + H$, and the left hand side of (6.21) is the mean square of $I : H$. The equation (6.21) expresses the fact that the mean of $X(u)$ diminishes, on the average, in absolute value as H increases, a fact naturally connected with the presence of zeros. An equation of this kind is, indeed, the kernel of the proof.

To carry out the details, however, we have to compare I and $\underset{\sim}{I}$. There is no known direct means of averaging $\underset{\sim}{I}$ (as opposed to $\underset{\sim}{I}^2$). Now $\underset{\sim}{I}$

is substantially $B\int_t^{t+H}|\zeta(\frac{1}{2}+iu)\,du$. We proceed in our proof by using the inequality

$$(6.22) \qquad \int_t^{t+H}|\zeta|\,dt \geqq \left|\int_t^{t+H}\zeta\,dt\right| \geqq \Re\int_t^{t+H}\zeta\,dt = H + \Re M;$$

and we show that, when $H > \frac{A}{\log T}$, the mean square of $|M|$ is $O(H)$, so that M is generally $O(\sqrt{H})$. This enables us to show that $\underset{\sim}{I}$ is generally greater than $\frac{1}{2}BH$, when H is a sufficiently large constant, and so, since I is generally less than $\frac{1}{2}BH$, to deduce our theorem.

But since $\sqrt{H}$ dominates H when H is small, the argument fails when H is a *small* constant, and this is the obstacle to further progress.

When H is small, $\Re M$ is (generally) more important than H. It might be supposed that the mean square of $B|\int\zeta\,dt|$ is greater than that of I, and that, if we could overcome difficulties of detail, we could conclude that $\underset{\sim}{I}$ is generally greater than I. But unfortunately the mean square of $B|\int\zeta\,dt|$, when H is small, is asymptotically *one half* that of I, as may be verified from (5.33)[10]. It would appear, then, that we lose something essential in replacing $\int|\zeta|\,dt$ by $|\int\zeta\,dt|$. This does not sound very surprising at first sight. But there is less difference between the two expressions, or between $|X(u)|$ and $B\zeta(\frac{1}{2}+iu)$, than might be supposed. If we assume the Riemann hypothesis, and write

$$N(T) = \frac{1}{2\pi}(T\log T - (1+\log 2\pi)T) + R(T),$$

we have, from (2.61),

$$|\zeta(\tfrac{1}{2}+it)| = A\,|X(t)|\left(1+O\left(\tfrac{1}{t}\right)\right)$$

$$\zeta(\tfrac{1}{2}+it) = A\,e^{iC}\,e^{\pi i R(t)}\,|X(t)|\left(1+O\left(\tfrac{1}{t}\right)\right).$$

Now it is known that $R(T) = o(\log T)$ and it is possible to show that

$$\frac{1}{T}\int_0^T |R(t)|\,dt = O(\log\log T).$$

Thus we should expect that, provided $H = o\left(\frac{1}{\log\log T}\right)$, $\left|\int_t^{t+H}\zeta(\frac{1}{2}+iu)\,du\right|$ would generally be asymptotically equivalent to $\int_t^{t+H}|\zeta(\frac{1}{2}+iu)|\,du$.

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[10]) It is easily shown that, when H is small,

$$4\sum^{\tau}\frac{\sin^2(\frac{1}{2}H\log n)}{n(\log n)^2} \sim 4\int_1^{\infty}\frac{\sin^2(\frac{1}{2}H\log u)}{u(\log u)^2}\,du = \pi H.$$

6.3. We note finally a deduction from Lemma 18 which, though we are unable to make any use of it, appears very curious.

Let K satisfy the same conditions as H. Then, since

$$I(t, H+K) = I(t, H) + I(t+H, K),$$

we have

$$\int_T^{T+U} I(t, H) I(t+H, K)\, dt = \tfrac{1}{2}\int_T^{T+U} I^2(t, H+K)\, dt - \tfrac{1}{2}\int_T^{T+U} I^2(t, H)\, dt$$
$$- \tfrac{1}{2}\int_T^{T+U} I^2(t+H, K)\, dt.$$

Now

$$\int_T^{T+U} I^2(t+H, K)\, dt = \int_T^{T+U} I^2(t, K)\, dt - \int_T^{T+H} I^2(t, K)\, dt + \int_{T+U}^{T+U+H} I^2(t, K)\, dt,$$

$$\int_T^{T+U} (I^2(t+H, K) - I^2(t, K))\, dt = O(HK^2 T^{\frac{1}{2}+\varepsilon}) = O\left(\frac{U}{\log T}\right),$$

provided c is small enough. Hence

$$\int_T^{T+U} I(t, H) I(t+H, K)\, dt = \frac{1}{2}\int_T^{T+U} I^2(t, H+K)\, dt - \frac{1}{2}\int_T^{T+U} I^2(t, H)\, dt$$
$$- \frac{1}{2}\int_T^{T+U} I^2(t, K)\, dt + O\left(\frac{U}{\log T}\right) = O\left(\frac{U}{\log T}\right),$$

by Lemma 18. This is true uniformly for $0 \leqq H \leqq T^c$, $0 \leqq K \leqq T^c$, when $c = c(a)$ is sufficiently small.

(Eingegangen am 14. Oktober 1920.)