

**CONJECTURE OF TWIN PRIMES (STILL
UNSOLVED PROBLEM IN NUMBER THEORY)
AN EXPOSITORY ESSAY**

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Abstract. The purpose of this paper is to gather as much results of advances, recent and previous works as possible concerning the oldest outstanding still unsolved problem in Number Theory (and the most elusive open problem in prime numbers) called "Twin primes conjecture" (8th problem of *David Hilbert*, stated in 1900) which has eluded many gifted mathematicians. This conjecture has been circulating for decades, even with the progress of contemporary technology that puts the whole world within our reach. So, simple to state, yet so hard to prove. Basic Concepts, many and varied topics regarding the Twin prime conjecture will be cover.



*Petronas towers (Twin towers)
Kuala Lumpur, Malaysia*

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1 Introduction

The prime numbers's study is the foundation and basic part of the oldest branches of mathematics so called "Arithmetic" which supposes the establishment of theorems. In Arithmetic, it is a matter of working only on known quantities, progressing step by step from the known to the unknown.

It is only since the arrival of computers in the 20th century that prime numbers have found a practical use. Although mathematicians (who often like challenges) have been interested since antiquity, prime numbers, the elementary bricks in the construction of the great edifice of natural numbers [29], continue to fascinate both the amateur and the professional. The prime numbers (the stars of mathematics) have been known for millennia but have not yet revealed all their secrets, even if this epoch is fertile for prime numbers. In fact, an amazing discovery found (in March 15, 2016) by *Kannan Soundararajan* and *Robert J. Lemke Oliver*, two researchers from *Stanford* University (*California*), which said (in their posted work: "Unexpected biases in the distribution of consecutive primes") that prime numbers would be much less random than was thought and could follow one another in a logical way [42, 50].

Among these pretty numbers, a special category called "twin primes", which represent an exclusive marvelous world of prime numbers's kingdom. The term "twin prime" was coined by the German mathematician *Paul Stäckel* (1862-1919) in the late 19th century. *P. Stäckel* carried out some numerical calculations in connection with this and related problems.

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The emphasis, in this paper, is on the most famous recent and previous works concerning the longest-standing still open "Twin primes conjecture".

My primary objective is to present the basic notions, theorems and results in a very simple way (without any complications) to any reader, beginner, specialist, amateur or professional in the field of Arithmetic.

For further results, properties, notes, theorems (and their proofs), characterizations, formulas and elementary criteria which address twin primes, the reader may consult an ample bibliography including many publications and original works, among which [1, 2, 13, 19, 28, 43, 45, 51, 57, 59, 60, 68, 71].

Definition 1. (*A twin prime*) [2, 21, 60, 66, 75]

A twin prime is a pair (p, q) such that both of the following hold:

- p and q are two odd consecutive natural numbers (thus p is 2 less than q).
- p and q are both prime numbers.

Definition 2. *The difference between two primes is called the "prime gap".*

Definition 3. [72]

Odd primes which are not in the set of twin primes are called "isolated primes".

Notation 4. *Throughout this paper, we let \mathcal{P} denote the set defined by*

$$\mathcal{P} = \left\{ (p, p+2) \in \mathbb{N}^2 : (p, p+2) \text{ is a twin prime} \right\}.$$

Conjecture 5. (*Twin primes conjecture*) [31, 40, 52, 56, 73, 75]

The twin primes conjecture (or *Euclid's* twin primes conjecture) posits that there are an infinite number of twin primes.

The Twin primes conjecture is still a mysterious open problem in Number Theory (it remains open as of late 2013 despite the simplicity of its statement). The most prominent mathematician of Greco-Roman antiquity *Euclid* (325 – 265 b.c.e., lived mainly in *Alexandria, Egypt*), gave the oldest extremely elegant proof (by contradiction) 2300 years ago (about 300 b.c.e., Theorem 20 in Book *IX* of *The Elements*) that there exist an infinite number of primes, there is no biggest prime [8, 13, 14, 16, 22, 29, 60, 62, 64, 69], and he conjectured that there are an infinite number of twin primes [21, 31], so the Twin primes conjecture is usually attributed to him.

2 History and some interesting deep results

In 1849 (the year he was admitted to Polytechnic), the Prince, French mathematician and artillery officer *Alphonse de Polignac* (1826–1863) made the more general conjecture (called "*Polignac's* conjecture") that for every natural number k , there are infinitely

many primes p such that $(p + 2k)$ is also prime [17, 38, 53, 56, 67].

The case $k = 1$ corresponds to the twin primes conjecture. Here is an interesting consequence concerning the abundance of twin primes [58]:

Corollary 6. *For every integer $m \geq 1$, there exist $2m$ consecutive primes which are m couples of twin primes.*

There is an obvious *Fermat* version of *Clement's* theorem (below), which we rarely cite in the literature [1]:

Theorem 7. (*Fermat characterization*)

If natural numbers $p, (p + 2)$ are both prime, then

$$2^{p+2} \equiv (3p + 8) \pmod{p(p + 2)}.$$

Theorem 7. can also be generalized to [1]:

Theorem 8. *The natural numbers $p, (p + 2)$ are both prime if and only if for all primes q such that: $q < p$, we have*

$$2q^{p+1} \equiv (p(q^2 - 1) + 2q^2) \pmod{p(p + 2)}.$$

In 1949, twin primes have been characterized by *Clement* as follows [1, 7, 9, 12, 25, 38, 46, 57, 58, 63, 78]

Proposition 9. (*P. A. Clement theorem, 1949*)

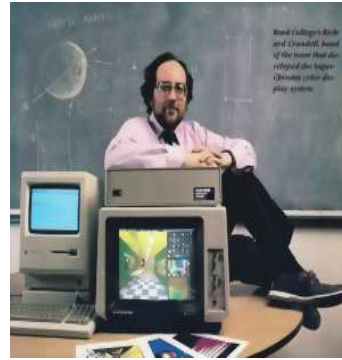
Let $p \geq 2$. The integers p and $(p + 2)$ form a pair of twin primes if and only if

$$\left(p + 4((p - 1)! + 1) \right) \equiv 0 \pmod{p(p + 2)}.$$

Clement gave a necessary and sufficient condition for twin primes. Although it has no practical value in the determination of twin primes.



Carl Bernard Pomerance
(Dartmouth College, Hanover)



Richard E. Crandall
Center for Advanced Computation
(Reed College)

The American physicist and computer (*Apple* distinguished) scientist *Richard E. Crandall* (1947-2012) (Director, Center for Advanced Computation, *Reed* College) and the American number theorist *Carl Bernard Pomerance* (born in 1944) of *Dartmouth* College, *Hanover*, (winner of the *Levi L. Conant* Prize in 2001) see this theorem (*Clement's* theorem) as "a way to connect the notion of twin-prime pairs with the *Wilson-Lagrange* theorem" [13].

Another interesting and intriguing result due to *I. S. A. Sergusov* (1971) and *W. G. Leavitt & A. A. Mullin* (1981) is [30, 37]

Theorem 10. (*Characterization of Sergusov, Leavitt and Mullin*)
 $(n \text{ is a product of a twin prime}) \iff (\phi(n) \cdot \sigma(n) = (n - 3)(n + 1))$

where: [2, 11, 20, 37, 43]

$\phi(n) = \text{card}(V)$ ($\phi(n)$ is called "Euler's totient function" or "Euler's ϕ -function"),

$V = \{k \in \mathbb{N} : 1 \leq k \leq n \text{ and } \text{gcd}(k, n) = 1\}$,

(if k belongs to V , it's called "a totative to n "), and

$\sigma(n)$ is the sum of the divisors of n (including 1 and n).

A number of important results about the spacing of prime numbers were been derived by *Franz Mertens* (1840-1927), a German mathematician of the late 19th and early 20th century [8, 76].

So, the Norwegian mathematician *Viggo Brun* (1885-1978) adapted and improved earlier work of *J. Merlin* (1911) entitled: "Sur Quelques Théorèmes d'Arithmétique et un Énoncé qui les Contient" [44]. *Brun* introduced modern sieves in 1920, in a paper titled "The Sieve of *Erastosthenes* and the theorem of *Goldbach*". He proved many results, among them [7, 52, 66, 77]: For sufficiently large x , the number of prime twins not exceeding x does not exceed $\frac{100x}{(\log x)^2}$.



Viggo Brun
(University of Oslo)

Theorem 11. (*Viggo Brun's theorem*, 1919) [49, 66, 77]

In number theory, *Brun's* remarkable and famous theorem (which came as a surprise) states that the sum of the reciprocals of the twin primes converges to a finite value known as *Brun's constant* which is equal to 1.9021605 (although the sum $\sum \frac{1}{p}$ over all primes diverges to infinity).

In explicit terms the sum [35, 55, 60]

$$\sum_{(p,p+2) \in \mathcal{P}} \left(\frac{1}{p} + \frac{1}{p+2} \right) = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \dots$$

either has finitely many terms or has infinitely many terms but is convergent: its value is known as *Brun's constant* [56].

Modern sieve methods originated with *Brun* around 1920. He used a new sieve to obtain several important number-theoretic results, notably an estimate of the density of twin primes [32, 35, 49]. *Brun's* constant was calculated by *Richard P. Brent* in 1976 as approximately 1.90216054 using the twin primes up to 100 billion [35, 55]. But this fact does not eliminate the possibility that there are infinitely many pairs of twin primes, it only shows that they are distributed infrequently among the real numbers [52]. Estimating *Brun's* Constant from 1942 to 2001 are resumed in [47]. *Brun* proved too that for every $m \geq 1$ there exist m successive primes which are not twin primes [57, 58].

Based on heuristic considerations about the distribution of twin primes, *Brun's* constant has been calculated, for example, by *Daniel Shanks* and *John W. Wrench, Jr.* (1974) [65], by *Richard P. Brent* (1976) [5], and more recently by *Thomas R. Nicely* (1995) (a computational number theorist of *Lynchburg* College, *Virginia*), who obtained:

Brun's constant=1.902160577783278...[48, 58, 75], thanks to an "INTEL" computer donation, and the help of several "INTEL" engineers in the development of *Nicely's*

numerical program. But *Nicely* had discovered a flaw (in *INTEL* Pentium) which led to erroneous values for certain division operations, what prompted "INTEL" (in 1995) to spend millions (475 millions) of dollars to repair this flaw [35, 59, 75]. More recently, *Pascal Sebah* (2002) showed that *Brun's* constant=1.902160583104, using all twin primes less than 10^{16} [75].



Jing-Run Chen

Joint research professor,
(Institute of Mathematics, Campus of *Tsinghua* University)

The best result to date, with sieve methods, is due to the Chinese mathematician *Jing-Run Chen* (1933-1996) (awarded the first rank of National Natural Science Prize, *He-Liang-He-Li* Prize and *Hua Lookeng* Mathematics Prize) [10].

Chen asserted and announced in 1966, published in 1973 and 1978 that there is an infinity of primes p such that $(p + 2)$ is either prime or the product of two primes (p is so called a "Chen prime" and $(p + 2)$ is so called a "semiprime"), this is very close to showing that there are infinitely many twin primes [32, 58, 73, 72, 77].

A proof of *Chen's* theorem is given in [27, 61].

Tony Forbes (a keen amateur number theorist) described an efficient integer squaring algorithm that was used on a 486 computers to discover (successfully) a large pair of twin primes [24].

In 2005, *Goldston*, *Pintz* and *Yildirim* introduced a method for studying small gaps between primes by using approximations to the prime k -tuples conjecture. This is now known as the "GPY method".

In 2010, *Peter A. Lindstrom* shows in a very interesting paper [39] how a given pair of twin primes can be used to generate other twin primes using *Goldbach's* conjecture.

An other interesting theorem concerning the existence and location of the twin primes may be found in [67].

2.1 *Yitang Zhang's discovery (April 17, 2013)*

On April 17, 2013 [6, 75]: the Chinese mathematician *Yitang "Tom" Zhang* (born in 1955) (a popular lecturer at the University of *New Hampshire*, which remained in the amber until April 2013) stunned the mathematical world (after many years of individual work) when he announced and proved a magnificent and beautiful result (called "the weak conjecture of twin prime numbers") (in *Zhang's* paper written with impeccable clarity and accepted by *Annals of Mathematics Journal* in early May 2013 [79]) which states that without any assumptions, there are infinitely many pairs of primes that differ by N , for some integer N that is less than 70 million [3, 6, 17, 18, 31, 40, 75, 79].



Yitang Zhang
(University of *New Hampshire*)

As said *E. Kowalski* in [36]:
" *Y. Zhang* proved in his article [79] the following theorem

Theorem 12. (*Zhang*)

There exists an even integer $h \geq 2$ with the property that there exist infinitely many pairs of prime numbers of the form $(p, p + h)$. In fact, there exists such an h with $h \leq 7 \times 10^7$.

Equivalently, if $p_n, n \geq 1$, denotes the n^{th} prime number, we have

$$\liminf_{n \rightarrow +\infty} (p_{n+1} - p_n) < +\infty,$$

and more precisely

$$\liminf_{n \rightarrow +\infty} (p_{n+1} - p_n) \leq 7 \times 10^7."$$

In other words [54], *Zhang* proved that there are infinitely many pairs of distinct primes (p, q) such that:

$$|p - q| < 7 \times 10^7.$$

This was a major step towards the celebrated twin prime conjecture [54].

Zhang made his fascinating extraordinary contribution, towards *Polignac's* conjecture, by the development of a private insight. While 70 million seems like a very big number, experts believe that it is only a matter of time before the number is drastically reduced [70].

2.2 "Polymath project"

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2.2.1 Computational successes (June 4, July 27, 2013)

In fact, the attempts to prove "Twin primes conjecture" has seen great advances, shortly after *Zhang's* contribution, by June 4, 2013, *Terence Tao*, an Australian mathematician (born in 1975) of the University of *California, Los Angeles*, which has a remarkable academic career marked by international awards (he was a winner of the *Fields Medal* (2006)) and prestigious publications, had created an exceptional internet based project called "Polymath8 project" (Polymath8a & Polymath8b), an open, online collaboration to find more accurate and better estimates for improving the bound that attracted dozens of participants (of highly collaborative attitude) [23].



Terence Tao
(University of *California*)

For some weeks, the obstinate project which brings together illustrious talented mathematicians moved forward at a breathless pace [75]. *Tao* played a major role in conducting and setting up the project, facilitating and verifying the arguments of each contributor [6].

"At times, the bound was going down every thirty minutes", *Tao* recalled.

By the end of May (2013), the bound had dropped to 42342946. As of July 20, 2013 the bound is a mere 5414 [75]. By July 27, 2013, the team had succeeded in reducing the proven bound on prime gaps from 70 million to 4680 (a rapid progress which is quite remarkable: in less than 4 months the bound is divided by more than 14000) [34].

2.2.2 Spectacular progress (November 19, 2013)

November 19, 2013, *James Maynard* (born in 1987) (awarded the *Sastra Ramanujan Prize* (2014), *Whitehead Prize* (2015) and *EMS Prize* (2016)), a British postdoctoral researcher working on his own at the University of *Montreal*, has passed the stage. Just few months after *Zhang* announced his result, *Maynard* who made a serious and intense activity (inspired by *Zhang's* work), has presented an independent proof that pushes the gap down to 600 [3, 79].



James Maynard
(University of *Montreal*)

The "Polymath8 project" tried to combine the collaboration's techniques with *Maynard's* approach to push this bound even lower [34].

The method of *Maynard* has the advantage of being flexible enough, and allows to show many things and often with a fairly comfortable uniformity [36]. This makes it possible, for example, to find several consecutive prime numbers in the same given arithmetic progression and very close to one another [41].

"The community is very excited by this new progress", *Tao* said [34].

From this effervescence, other ideas might still arise to demonstrate the conjecture of the twin primes.

In (2014) [54], the Indo-Canadian mathematician *Maruti Ram Pedaprolu Murty* (born in 1953), head of the Department of Mathematics and Statistics at *Queen's University* and winner of numerous prestigious awards in mathematics, including *Coxeter–James Prize* (1988), discussed greater details and outlined proofs of these later works of *Zhang, Tao and Maynard*.

Recently, as of April 14, 2014, one year after *Zhang's* announcement, the bound has been reduced to 246 using the same methods of *Maynard* and *Tao* and a simpler approach [3]. Now, the mathematicians are working to narrow that difference down to just 2. The cooperative team "Polymath8" did not yet say his last word.

3 Some of largest (titanic & gigantic) known twin primes

Definition 13. A "titan", as defined (in 1984) by the American computer engineer and mathematician *Samuel Yates* (1919–1991), means any person who has discovered a titanic prime number (any prime with 1000 or more decimal digits).

And in 1992, the term "gigantic" (any prime with 10000 or more decimal digits) appeared (after the death of *S. Yates*) in *Journal of Recreational Mathematics* in the article "Collecting gigantic and titanic primes" (1992) by *Samuel Yates and Chris Caldwell*.

In the same way, "megaprimes" are primes with at least a million digits.

In this section, I will give some "titanic" and other "gigantic" twin primes, their discoverers, year of discovery, and number of digits of each twin primes:

- In 1989, *B. K. Parady, J. F. Smith* and *S. Zarantonello* found the titanic twin primes: $663777 \times 2^{7650} \pm 1$. They have 2309 decimal digits [26].
- In 1990, *B. K. Parady, J. F. Smith* and *S. Zarantonello* discovered the titanic twin primes: $1706595 \times 2^{11235} \pm 1$ [55, 72].
- In 1991, *H. Dubner* found the titanic twin primes: $1171452282 \times 10^{2490} \pm 1$. They have 2500 decimal digits [47].
- In 1993, *H. Dubner* found the titanic twin primes: $459 \times 2^{8529} \pm 1$. They have 2571 decimal digits [47].

- In 1994, *K.-H. Indlekofer* and *A. Jàrai* found the pair $697053813 \times 2^{16352} \pm 1$. They have 4932 decimal digits [47, 57].
- In 1995, *Harvey Dubner* discovered the largest known pair of twin primes: $570918348 \times 2^{5120} \pm 1$. They have 5129 decimal digits [47, 57].
- In 1998, largest findings being the pair (found by *R. Ballinger* and *Y. Gallot*): $835335 \times 2^{39014} \pm 1$. They have 11751 decimal digits [15, 13].
- In 1999, largest findings being the pair (found by *H. Lifchitz*): $361700055 \times 2^{39020} \pm 1$ [13].
- Also, in 1999, largest known twin primes found by *Eric Vautier* was: $2003663613 \times 2^{195000} \pm 1$. These primes have 58711 decimal digits [28].
- In 2000, the gargantuan twin prime pair found (by *H. Wassing*, *A. Jàrai* and *K.-H. Indlekofer*) was: $2409110779845 \times 2^{60000} \pm 1$ [13].

For further values of largest known twin primes in 2000, 2001 and 2002, the reader can consult [35, 58].

- In 2005, the largest known twin primes found by *A. Jàrai et al.* was $16869987339975 \times 2^{171960} \pm 1$. They have 51779 decimal digits [35].
- In 2006, the largest known twin primes found was $100314512544015 \times 2^{171960} \pm 1$. They have 51780 decimal digits [74].
- The largest known twin prime found in January 15, 2007 by *Eric Vautier* was: $2003663613 \times 2^{195000} \pm 1$. These primes have 58711 decimal digits [33].
- In 2009, the gigantic known twin prime pair discovered (by two distributed computing projects: *Twin prime search* and *PrimeGrid*) was $65516468355 \times 2^{333333} \pm 1$. These primes have 100355 decimal digits [33].
- In December 25, 2011, the gigantic known twin prime numbers discovered as part of the *PrimeGrid distributed calculation* project were: $3756801695685 \times 2^{666669} + 1$ and $3756801695685 \times 2^{666669} - 1$. They have 200700 decimal digits [75].
- The gigantic known twin prime numbers found (on September 14, 2016) were: $2996863034895 \times 2^{1290000} + 1$ and $2996863034895 \times 2^{1290000} - 1$. They have 388342 decimal digits [4].

The reader may find an extensive list of largest known twin primes in [78].

4 Properties

Proposition 14. *Every twin prime pair except $(3, 5)$ is of the form $(6n-1, 6n+1)$ for some natural number n . (Thus, the number between the two first (second respectively) components of twin prime pairs is a multiple of 6).*

Proof. Let $(p, p+2)$ be a twin prime such that $(p, p+2) \neq (3, 5)$, thus $p \geq 5$ and p is a prime number. So, p is a natural number, hence it must be written in the form: $p = 6n + \ell$, where $\ell \in \{-1, 0, 1, 2, 3, 4\}$ and $n \in \mathbb{N}^*$.

The assumption (p is prime) rules out the cases $\ell \in \{0, 2, 3, 4\}$, since if so, then $p = 6n$ or $p = 6n + 2 = 2(3n + 1)$ or $p = 6n + 3 = 3(2n + 1)$ or $p = 6n + 4 = 2(3n + 2)$, and thus p would be composite, hence p is not prime (which is false).

Therefore, it remains only two possible cases: $\ell \in \{-1, 1\}$ and this means that p is (necessarily) a prime of the form $p = 6n \pm 1$ such that $n \in \mathbb{N}^*$. \square

Corollary 15. *All prime numbers, except 2 and 3, are either of the form $6n - 1$ or of the form $6n + 1$ for some natural number n .*

Corollary 16. *The gcd: "great common divisor" of twin primes is 1.*

Proposition 17. *Let $(6n - 1, 6n + 1)$ be a twin prime pair such that: $n \in \mathbb{N}^* - \{1\}$, thus*

- *The last digit of $6n + 1$ is necessarily different from 7.*
- *The last digit of $6n - 1$ is necessarily different from 3.*

Proof. Since $n \in \mathbb{N}^* - \{1\}$, then

- It yields $6n + 1$ is a prime number, therefore it is odd and it's last digit can't be 7, because if so, then the last digit of the prime number $6n - 1$ is equal to 5 (thus, $6n - 1$ is divisible by 5), which is a contradiction (since $6n - 1$ is prime). We deduce from all this that the last digit of $6n + 1$ is necessarily different from 7.
- Arguing analogously, it yields $6n - 1$ is a prime number, therefore it is odd and it's last digit can't be 3, because if so, then the last digit of the prime number $6n + 1$ is equal to 5 (thus, $6n + 1$ is divisible by 5), which is a contradiction (since $6n + 1$ is prime). We deduce from all this that the last digit of $6n - 1$ is necessarily different from 3.

\square

Definition 18. • *The digit sum of a natural number k is the value obtained by summing its digits.*

- The digital root of a natural number k is the (single digit) value obtained by repeating the digital sum process (an iterative process). So, the digital root of a natural number k is the remainder obtained by dividing k by 9.

Example 19. The digit sum of 36 is: $3 + 6 = 9$.

The digital root of 369875 is obtaining as follows:

The first digit sum of 369875 is: $3 + 6 + 9 + 8 + 7 + 5 = 38$.

The second digit sum of 369875 is: $3 + 8 = 11$.

The third digit sum of 369875 is: $1 + 1 = 2$.

Thus, the digital root of 369875 is 2.

Proposition 20. [35]

- The digital root of the product of 3 and 5 (twin primes) is 6.
- The digital root of the product of twin primes, other than 3 and 5, is 8.

Proof. We have:

- The product of 3 and 5 (twin primes) is equal to 15, but $1 + 5 \equiv 6 \pmod{9}$.
- Let $(6n - 1, 6n + 1)$ be a twin prime pair such that: $n \in \mathbb{N}^* - \{1\}$, thus, the product $(6n - 1)(6n + 1)$ verify

$$\begin{aligned} (6n - 1)(6n + 1) &= 36n^2 - 1 \\ &\equiv (0 - 1) \pmod{9} \\ &\equiv (9 - 1) \pmod{9} \\ &\equiv 8 \pmod{9} \end{aligned}$$

□

5 First twin primes less than 3002

Table 1. contains explicitly the twin primes less than 1000 (they are 35).

Table 2. contains explicitly the twin primes less than 2000 and bigger than 1000 (they are 25).

Table 3. contains explicitly the twin primes less than 3002 and bigger than 2000 (they are 21).

Remark 21. Look at the following tables: Table 1, Table 2 and Table 3, the reader can observe that for each twin prime pair and for each $n \in \mathbb{N}^* - \{1\}$, the last digit of $6n - 1$ is always 1 or 7 or 9 whereas the last digit of $6n + 1$ is always 3 or 9 or 1.

Twin primes pairs

Rank	p	q	$(p, q) = (6n - 1, 6n + 1)$	n
0	3	5	(3, 5)	/
1	5	7	(5, 7)	1
2	11	13	(11, 13)	2
3	17	19	(17, 19)	3
4	29	31	(29, 31)	5
5	41	43	(41, 43)	7
6	59	61	(59, 61)	10
7	71	73	(71, 73)	12
8	101	103	(101, 103)	17
9	107	109	(107, 109)	18
10	137	139	(137, 139)	23
11	149	151	(149, 151)	25
12	179	181	(179, 181)	30
13	191	193	(191, 193)	32
14	197	199	(197, 199)	33
15	227	229	(227, 229)	36
16	239	241	(239, 241)	40
17	269	271	(269, 271)	45
18	281	283	(281, 283)	47
19	311	313	(311, 313)	52
20	347	349	(347, 349)	56
21	419	421	(419, 421)	70
22	431	433	(431, 433)	72
23	461	463	(461, 463)	77
24	521	523	(521, 523)	87
25	569	571	(569, 571)	95
26	599	601	(599, 601)	100
27	617	619	(617, 619)	103
28	641	643	(641, 643)	107
29	659	661	(659, 661)	110
30	809	811	(809, 811)	135
31	821	823	(821, 823)	137
32	827	829	(827, 829)	138
33	857	859	(857, 859)	143
34	881	883	(881, 883)	147

Table 1: The first twin primes less than 1000

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Twin primes pairs

Rank	p	q	$(p, q) = (6n - 1, 6n + 1)$	n
35	1019	1021	(1019, 1021)	170
36	1031	1033	(1031, 1033)	172
37	1049	1051	(1049, 1051)	175
38	1061	1063	(1061, 1063)	177
39	1091	1093	(1091, 1093)	182
40	1151	1153	(1151, 1153)	192
41	1229	1231	(1229, 1231)	205
42	1277	1279	(1277, 1279)	213
43	1289	1291	(1289, 1291)	215
44	1301	1303	(1301, 1303)	217
45	1319	1321	(1319, 1321)	220
46	1427	1429	(1427, 1429)	238
47	1451	1453	(1451, 1453)	242
48	1481	1483	(1481, 1483)	247
49	1487	1489	(1487, 1489)	248
50	1607	1609	(1607, 1609)	268
51	1619	1621	(1619, 1621)	270
52	1667	1669	(1667, 1669)	278
53	1697	1699	(1697, 1699)	283
54	1721	1723	(1721, 1723)	287
55	1787	1789	(1787, 1789)	298
56	1871	1873	(1871, 1873)	312
57	1877	1879	(1877, 1879)	313
58	1931	1933	(1931, 1933)	322
59	1949	1951	(1949, 1951)	325
60	1997	1999	(1997, 1999)	333

Table 2: The twin primes less than 2000 and bigger than 1000

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Twin primes pairs				
Rank	p	q	$(p, q) = (6n - 1, 6n + 1)$	n
61	2027	2029	(2027, 2029)	338
62	2081	2083	(2081, 2083)	347
63	2087	2089	(2087, 2089)	348
64	2111	2113	(2111, 2113)	352
65	2129	2131	(2129, 2131)	355
66	2141	2143	(2141, 2143)	357
67	2237	2239	(2237, 2239)	373
68	2267	2269	(2267, 2269)	378
69	2309	2311	(2309, 2311)	385
70	2339	2341	(2339, 2341)	390
71	2381	2383	(2381, 2383)	397
72	2549	2551	(2549, 2551)	425
73	2591	2593	(2591, 2593)	432
74	2657	2659	(2657, 2659)	443
75	2687	2689	(2687, 2689)	448
76	2711	2713	(2711, 2713)	452
77	2729	2731	(2729, 2731)	455
78	2789	2791	(2789, 2791)	465
79	2801	2803	(2801, 2803)	467
80	2969	2971	(2969, 2971)	495
81	2999	3001	(2999, 3001)	500

Table 3: The twin primes less than 3002 and bigger than 2000

6 Rarefaction of twin prime numbers

The proportion of twin primes among the prime numbers is decreasing. Table 4 shows the percentage of twin primes among all primes. Also, it contains the number of twin primes less than N for $10 \leq N \leq 10^{18}$ [15, 58].

N	Counts of twin primes	Counts of prime numbers	% of twin primes
10	2	4	50
10^2	8	25	32
10^3	35	168	20.83
10^4	205	1229	16.68
10^5	1224	9592	12.76
10^6	8169	78498	10.41
10^7	58980	664579	8.87
10^8	440312	5761455	7.65
10^9	3424506	50847534	6.73
10^{10}	27412679	455052511	6.02
10^{11}	224376048	4118054813	5.45
10^{12}	1870585220	37607912018	4.97
10^{13}	15834664872	346065536839	4.56
10^{14}	135780321665	3204941750802	4.22
10^{15}	1177209242304	29844570422669	3.94
10^{16}	10304195697298	279238341033925	3.69
10^{17}	90948839353159	2623557157654233	3.46
10^{18}	808675888577436	24739954287740860	3.26

Table 4: The number of twin primes less than N : $10 \leq N \leq 10^{18}$

The next histogram (Figure 1) represents the behaviour (the decreasing of the density) of the percentage of twin primes number versus the number of prime numbers.

Therefore, twin primes become increasingly rare as higher numbers are examined. Namely, twin primes are rarefying as one advances in the sequence of natural numbers.

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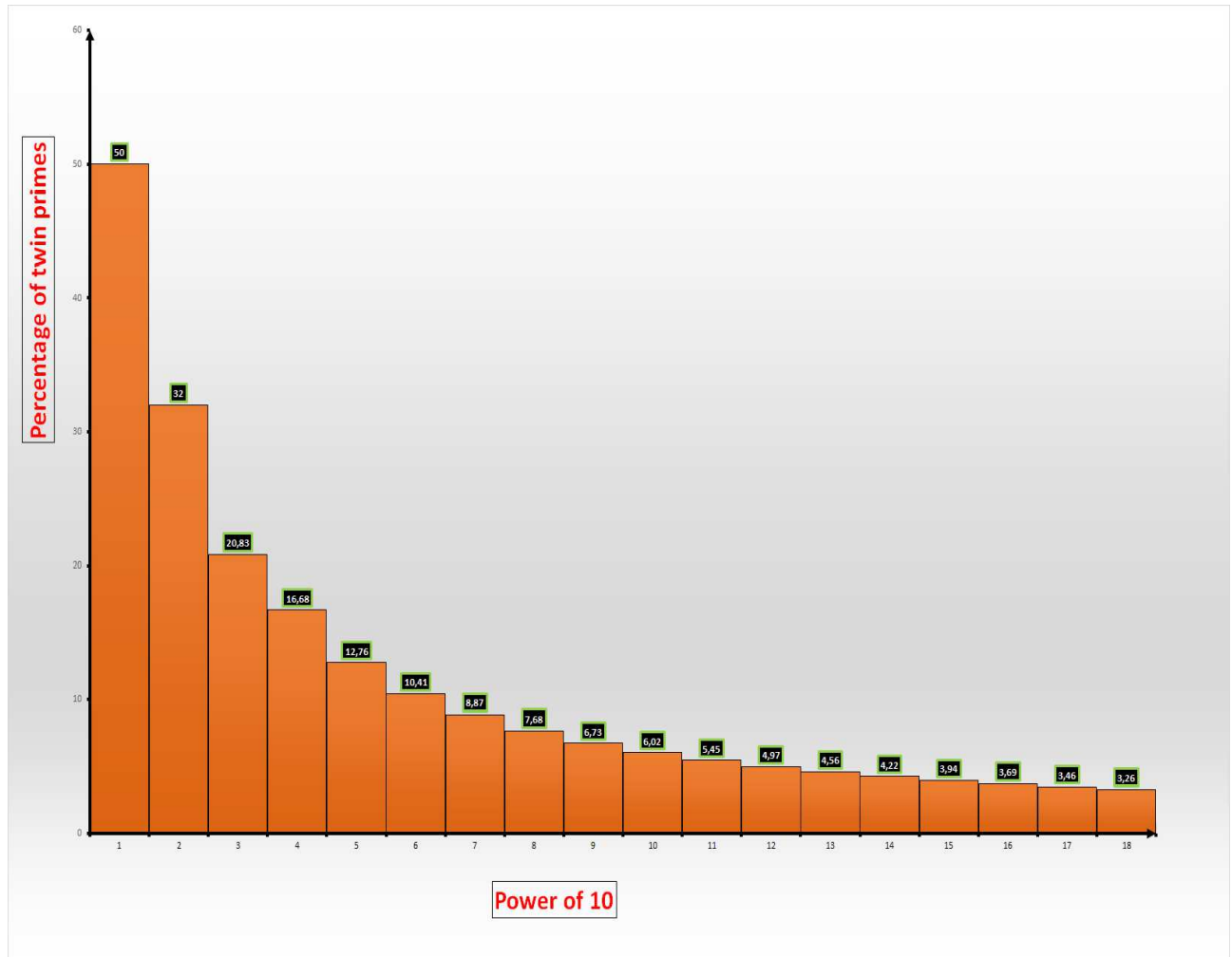


Figure 1: Proportion of twin primes among prime numbers

7 Conclusion

This paper cited names of several stellar, talented and great, ancient and modern mathematicians who have tried, far or near, to solve the enigma called "Twin primes conjecture".

In this work, I have outlined many properties, results, theorems relating to this conjecture. I hope that the reader of this paper will be enlightened, even if the path leading to a demonstration of "Twin primes conjecture" seems still very long. Despite its apparent simplicity, guesses are one thing, but proof remains the rewarding ultimate goal.

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